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Dimensionally continued infinite reduction of couplings

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Abstract: The infinite reduction of couplings is a tool to consistently renormalize a wide class of non-renormalizable theories with a reduced, eventually finite, set of independent couplings, and classify the non-renormalizable interactions. Several properties of the reduction of couplings, both in renormalizable and non-renormalizable theories, can be better appreciated working at the regularized level, using the dimensional-regularization technique. We show that, when suitable invertibility conditions are fulfilled, the reduction follows uniquely from the requirement that both the bare and renormalized reduction relations be analytic in $\varepsilon = D - d$, where D and d are the physical and continued spacetime dimensions, respectively. In practice, physically independent interactions are distinguished by relatively non-integer powers of ε. We discuss the main physical and mathematical properties of this criterion for the reduction and compare it with other equivalent criteria. The leading-log approximation is solved explicitly and contains sufficient information for the existence and uniqueness of the reduction to all orders.

Keywords: Renormalization Group, Renormalization Regularization and Renormalons.

Contents

1. Introduction

Formulated in the common fashion, non-renormalizable theories can be used only as low-energy effective field theories, due to the presence of infinitely many independent couplings, introduced to subtract the divergences. Although effective field theories are good for most practical purposes, they are unable to suggest new physics beyond them. The search for more fundamental theories that include non-renormalizable interactions can have applications both to physics beyond the Standard Model and quantum gravity. Moreover, it can clarify if quantum field theory is really inadequate, as some physicists think, to explain the fundamental interactions of nature beyond power-counting renormalizability. In that case, it can suggest a more fundamental theoretical framework to supersede quantum field theory.

The main tool to classify the irrelevant interactions is the "infinite reduction", that is to say the reduction of couplings in non-renormalizable theories. The reduction of couplings was first studied by Zimmermann [1, 2] for power-counting renormalizable theories, as an alternative to unification. The idea is to look for relations among the coupling constants that are consistent with the renormalization of divergences. Such relations, rather than being based on symmetries, follow from constraints on the solutions to the renormalization-group (RG) equations. The constraint proposed by Zimmermann is that the reduction should be analytic. Assume that a theory containing two couplings α and λ is an effective formulation of a more fundamental theory containing only the coupling α. It is clear that if the connection between the more fundamental theory and the effective theory is perturbative, then the relation $\lambda(\alpha)$ should be analytic in α, because a perturbative expansion cannot generate fractional, irrational, complex or negative-integer powers of α.

A non-renormalizable interaction is made of infinitely many lagrangian terms. The infinite reduction is the search for special relations that express the irrelevant couplings as unique functions of a reduced, eventually finite, set of independent couplings $\overline{\lambda}$, such that the divergences are removed by means of field redefinitions plus renormalization constants for the $\overline{\lambda}$s. It can be easily shown that in non-renormalizable theories the analyticity requirement proposed by Zimmermann is too strong. The basic reason is that an irrelevant deformation has no end, so it is impossible to identify the minimum power of a coupling, which is necessary to demand analyticity. Then it is natural to allow also for negative powers of the couplings and replace analyticity by perturbative meromorphy. That means that the reduction relations have to be meromorphic in the marginal couplings α, and that the maximum negative α-powers should be bounded by the order of the perturbative expansion [3].

Although perturbative meromorphy is an exhaustive criterion for the infinite reduction, it is useful to build a framework where the reduction is more natural and its properties can be more clearly appreciated. For this purpose, it is convenient to study the reduction of couplings at the regularized level, using the dimensional-regularization technique. The reduction of couplings is by construction RG invariant, so in a common regularization framework the bare and renormalized reduction relations look the same, because the bare couplings are nothing but the renormalized couplings calculated at an energy scale equal to the cut-off. However, the dimensional-regularization technique makes systematic use of the parameter $\varepsilon = D - d$, D and d being the physical and continued spacetime dimensions, respectively, and various properties can be conveniently formulated in terms of analyticity and meromorphy in the parameter ε.

We give two equivalent criteria for the infinite reduction. The first criterion is obtained studying the analyticity properties of the renormalized reduction relations and is an immediate generalization of the criterion formulated in ref. [3] at $\varepsilon = 0$. The second criterion is new, obtained comparing the analyticity properties of the renormalized and bare reduction relations.

If the theory contains a unique strictly-renormalizable coupling α, when certain invertibility conditions hold, the reduction is uniquely determined by the following two equivalent requirements:

1. that the renormalized reduction relations be perturbatively meromorphic in α at $\alpha = 0$ and analytic in ε at $\varepsilon = 0$;

2. that the renormalized and bare reduction relations be analytic in ε at $\varepsilon = 0$.

Requirement 2) will be called *double analyticity in ε*. Suitable generalizations of the criteria just stated apply to the theories that contain more marginal couplings.

Because of meromorphy in α, the strictly-renormalizable subsector of the theory must be fully interacting.

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The non-renormalizable interactions can be studied expanding the correlation functions perturbatively in the overall energy E, for $E \ll M_{P\mathrm{eff}}$, where $M_{P\mathrm{eff}}$ is an "effective Planck mass".

It is important to emphasize that in our analysis all super-renormalizable parameters, in particular the masses, are switched off, which is consistent in appropriate subtraction schemes, including the dimensional-regularization technique. This assumption ensures that the beta functions of the non-renormalizable couplings are polynomial in the non-renormalizable couplings themselves allowing an explicit solution of the reduction equations by a simple recursive procedure. The super-renormalizable parameters can be eventually incorporated in a second stage perturbatively in the reduction equations.

Now we illustrate the main features of the infinite reduction at the regularized level. Let α and λ denote two independent couplings. In general, the bare and renormalized reduction relations have the form

$$\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}(\alpha_{\mathrm{B}}, \zeta, \varepsilon), \qquad \lambda = \lambda(\alpha, \xi, \varepsilon), \tag{1.1}$$

respectively, where ζ and ξ are certain arbitrary constants, defined so that (1.1) are analytic in ε, ζ and ξ. Matching the relations (1.1) with each other, formulas

$$\zeta = \zeta(\xi, \varepsilon), \qquad \Leftrightarrow \qquad \xi = \xi(\zeta, \varepsilon) \tag{1.2}$$

relating the constants ζ and ξ can be worked out. In general, the relations (1.2) are not analytic in ε at $\varepsilon = 0$, which implies that the bare and renormalized relations are not contemporarily analytic in ε, for generic values of ζ and ξ. On the other hand, when certain invertibility conditions are fulfilled, there exists unique values $\overline{\zeta}$ and $\overline{\xi}$ such that both $\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}(\alpha_{\mathrm{B}}, \overline{\zeta}, \varepsilon)$ and $\lambda = \lambda(\alpha, \overline{\xi}, \varepsilon)$ are analytic in ε. Those values select the infinite reduction.

Independent irrelevant interactions are distinguished by relatively fractional, irrational, complex or negative-integer powers of ε. These properties provide a criterion to classify the non-renormalizable interactions.

The criterion just formulated can be justified as follows. The quantum action $\Gamma[\Phi, \alpha, \lambda, \varepsilon]$ is convergent in the physical limit, that is the limit $\varepsilon \to 0$ at fixed renormalized fields and couplings. If a reduction $\lambda\,(\alpha, \varepsilon)$ is consistent, then also the reduced quantum action $\Gamma[\Phi, \alpha, \lambda(\alpha, \varepsilon), \varepsilon]$ should be convergent in the physical limit, so $\lambda\,(\alpha, \varepsilon)$ should be regular for $\varepsilon \sim 0$. The bare lagrangian $\mathcal{L}(\varphi_{\mathrm{B}}, \lambda_{\mathrm{B}}, \alpha_{\mathrm{B}}, \varepsilon)$, on the other hand, tends to the classical lagrangian in the "naive" limit, that is the limit $\varepsilon \to 0$ at fixed bare fields and couplings. If a reduction $\lambda_{\mathrm{B}}\,(\alpha_{\mathrm{B}}, \varepsilon)$ is consistent, then the reduced bare lagrangian $\mathcal{L}(\varphi_{\mathrm{B}}, \alpha_{\mathrm{B}}, \lambda_{\mathrm{B}}(\alpha_{\mathrm{B}}, \varepsilon), \varepsilon)$ should converge to the reduced classical lagrangian in the naive limit, so $\lambda_{\mathrm{B}}(\alpha_{\mathrm{B}}, \varepsilon)$ should be regular for $\varepsilon \sim 0$. In perturbation theory it is safe to replace the words "convergent" and "regular" with the word "analytic". Indeed, the more fundamental theory is certainly analytic in ε and if the connection between the more fundamental theory and the effective theory is perturbative the bare and renormalized relations should be both analytic in ε.

In ref. [3] the infinite reduction was studied at $\varepsilon = 0$, where the relations among the couplings are uniquely selected by perturbative meromorphy. We will show that the infinite reduction at $\varepsilon \neq 0$, implied by the double ε-analyticity, gives results that are physically consistent with those of ref. [3]. The infinite reduction at $\varepsilon \neq 0$ has already been studied in [4] for a special class of models. It is worth to recall that when the renormalizable sector is an interacting conformal field theory the infinite reduction has peculiar features that make it conceptually simpler [5].

The study of quantum field theory beyond power counting has been attracting interest for a long time, motivated by effective field theory, low-energy QCD, quantum gravity and the search for new physics beyond the Standard Model. Here we are not concerned with the predictiveness of our theories. Our purpose is merely to give mathematical tools to organize and classify the non-renormalizable interactions, using analyticity properties and consistency with the renormalization group.

Sometimes *ad hoc* subtractions are used to make non-renormalizable theories predictive. Often they amount to assign preferred values (typically, zero) to an infinity of renormalized couplings at the subtraction point μ. Another example of ad hoc prescription is the BPZH subtraction [6] of divergent diagrams at zero momentum. These prescriptions, although appealing for a variety of reasons, are not consistent with the renormalization group. Our construction, on the other hand, follows precisely from consistency with the renormalization group.

A different approach to reduce the number of independent couplings in non-renormalizable theories is Weinberg's asymptotic safety [7], which has been recently studied using the exact renormalization-group techniques [8 − 10]. Other investigations of reductions of couplings in non-renormalizable theories have been performed by Atance and Cortes [11, 12], Kubo and Nunami [13], Halpern and Huang [14]. For a recent perturbative renormalization-group approach to non-renormalizable theories, see [15].

The paper is organized as follows. In section 2 we study Zimmermann's reduction of couplings at $\varepsilon \neq 0$ and exhibit an interesting connection with the ε-expansion techniques. In sections 3 and 4 we study the reduction in non-renormalizable theories, first at the regularized level and then at the bare level. In section 5 we study the physical invertibility conditions in the absence of three-leg marginal vertices, while in section 6 we explicitly solve the infinite reduction in the leading-log approximation, which contains enough information about the existence and uniqueness of the reduction to all orders. In most of the paper we assume that the renormalizable subsector contains a single strictly-renormalizable coupling. In section 7 we generalize our results to theories whose renormalizable subsector contains more independent couplings. Section 8 contains the conclusions, while the appendix contains the derivation of multivariable renormalization constants from the associated beta functions, which is used in the paper. We work in the euclidean framework.

2. Zimmermann's reduction at $\varepsilon \neq 0$

Consider a renormalizable theory with two marginal couplings, ρ and g, such as massless scalar electrodynamics,

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + |D_\mu\varphi|^2 + \frac{\rho}{4}(\overline{\varphi}\varphi)^2, \tag{2.1}$$

where $D_\mu\varphi = \partial_\mu\varphi + igA_\mu\varphi$, or the massless Yukawa model

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + \overline{\psi}\partial\!\!\!/\psi + g\varphi\overline{\psi}\psi + \frac{\rho}{4!}\varphi^4.$$

Define $\alpha = g^2$. Write the bare couplings and renormalization constants as

$$\alpha_{\rm B} = \mu^\varepsilon \alpha Z'_\alpha(\alpha,\rho,\varepsilon), \qquad \rho_{\rm B} = \mu^\varepsilon\left(\rho + \Delta_\rho(\alpha,\rho,\varepsilon)\right),$$

where Z'_α and Δ_ρ are analytic functions of the couplings. It is convenient to write $\rho = \alpha\eta$ and use α and η as independent couplings. Then a Feynman diagram carries a power of α equal to $v_4 + v_3/2$, where v_3, v_4 denote the numbers of three-leg and four-leg vertices, respectively. Using $4v_4 + 3v_3 = E + 2I$ and $V = v_4 + v_3 = I - L + 1$, where E, L, I and V denote the numbers of external legs, loops, internal legs and vertices, respectively, we have

$$v_4 + \frac{v_3}{2} = \frac{E}{2} - 1 + L. \tag{2.2}$$

Therefore, the counterterms that renormalize the three-leg vertex are proportional to $g\alpha^L$ and those that renormalize the four-leg vertex are proportional to α^{L+1}. Then it is possible to write

$$\alpha_{\rm B} = \mu^\varepsilon \alpha Z_\alpha(\alpha,\eta,\varepsilon), \qquad \eta_{\rm B} = \eta + \alpha\Delta_\eta(\alpha,\eta,\varepsilon), \tag{2.3}$$

and Z_α, Δ_η are analytic functions of α, η.

Reduction.

The reduction is a relation $\eta(\alpha)$ between the two couplings, such that a single renormalization constant, the one of α, is sufficient to remove the divergences associated with both α and η. This goal can be achieved imposing consistency with the renormalization group. In the minimal subtraction scheme we have

$$\frac{\mathrm{d}\eta}{\mathrm{d}\ln\mu} = \beta_\eta(\eta,\alpha), \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu} = \beta_\alpha(\eta,\alpha) - \varepsilon\alpha. \tag{2.4}$$

Because of (2.2) and (2.3), the beta functions have expansions

$$\beta_\alpha = \beta_1\alpha^2 + \sum_{k=2}^{\infty}\alpha^{k+1}P_k(\eta), \qquad \beta_\eta = \alpha\left(a + b\eta + c\eta^2\right) + \sum_{k=2}^{\infty}\alpha^k Q_{k+1}(\eta), \tag{2.5}$$

where $P_k(\eta)$ and $Q_k(\eta)$ are polynomials of order k. The one-loop structure of β_α follows from explicit analysis of the one-loop diagrams. Consistency with the RG equations implies the differential equation

$$\frac{\mathrm{d}\eta(\alpha)}{\mathrm{d}\alpha} = \frac{\beta_\eta(\eta(\alpha),\alpha)}{\beta_\alpha(\eta(\alpha),\alpha) - \varepsilon\alpha}, \tag{2.6}$$

that determines the solution $\eta(\alpha)$ up to an arbitrary constant ξ, the initial condition. If $\beta_\alpha \neq 0$ the solution has a smooth limit for $\varepsilon \to 0$, which is the Zimmermann solution [1].

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Reduction at the level of bare couplings.

The bare relations are quite simpler. Indeed, since η_{B} is dimensionless at $\varepsilon \neq 0$, while α_{B} is dimensionful, η_{B} is just a constant, so

$$\eta_{\mathrm{B}}(\alpha_{\mathrm{B}}, \xi, \varepsilon) = \zeta. \tag{2.7}$$

Re-written in terms of renormalized couplings, formula (2.7) gives

$$\eta + \alpha\Delta_\eta(\alpha, \eta, \varepsilon) = \eta + \sum_{k=1}^{\infty} \alpha^k \widetilde{P}_{k+1}(\eta, \varepsilon) = \zeta, \tag{2.8}$$

where $\widetilde{P}_{k+1}(\eta, \varepsilon)$ is a polynomial in η of degree $k+1$ and a divergent function of ε. Formula (2.8) is an algebraic equation for $\eta(\alpha)$. The solution coincides with the solution of (2.6) once the constants ξ and ζ are appropriately related to each other. Taking the limit $\alpha \to 0$ of (2.8), we obtain the relation between ζ and ξ, which reads

$$\zeta(\varepsilon, \xi) = \lim_{\alpha \to 0} \eta(\alpha, \xi, \varepsilon), \qquad \varepsilon \neq 0. \tag{2.9}$$

Using (2.9), equation (2.8) can be solved for $\eta(\alpha, \xi, \varepsilon)$ as an expansion in powers of α. Thus, the solution of the reduction equation is analytic in α at $\varepsilon \neq 0$.

Reduction of the renormalization constants.

Once η is written in terms of α and the arbitrary constant ξ, the divergences are removed with a single renormalization constant. To show this fact, it is convenient to work in an regularization framework (for example the cut-off method or the Pauli-Villars regularization) where the bare couplings α_{B} and η_{B} coincide with the renormalized couplings α_Λ and η_Λ at the cut-off scale Λ. In such a framework, using the minimal subtraction scheme, the RG consistency conditions are unaffected by the regularization, namely they have the same form as (2.6) at $\varepsilon = 0$. Denote the renormalization constants in this framework with $\widehat{Z}_\alpha(\alpha, \eta, \ln \Lambda/\mu)$ and $\widehat{\Delta}_\eta(\alpha, \eta, \ln \Lambda/\mu)$. Then, because of the RG consistency conditions, the relation $\eta = \eta(\alpha)$ holds at every energy scale, in particular Λ and the renormalization point μ. We have

$$\eta_{\mathrm{B}} = \eta(\alpha) + \alpha\widehat{\Delta}_\eta(\alpha, \eta(\alpha), \ln \Lambda/\mu) = \eta_\Lambda = \eta(\alpha_\Lambda) = \eta\left(\alpha\widehat{Z}_\alpha(\alpha, \eta(\alpha), \ln \Lambda/\mu)\right), \tag{2.10}$$

and so the renormalization constant of η is not independent, but uniquely related to the one of α:

$$\widehat{\Delta}_\eta(\alpha, \eta(\alpha), \ln \Lambda/\mu) = \frac{1}{\alpha}\left[\eta\left(\alpha\widehat{Z}_\alpha(\alpha, \eta(\alpha), \ln \Lambda/\mu)\right) - \eta(\alpha)\right]. \tag{2.11}$$

Formula (2.10) ensures that it is sufficient to renormalize the coupling α *inside* the function $\eta(\alpha)$ to remove the divergences associated with η. In the dimensional-regularization framework a relation between Δ_η and Z_α analogous to (2.11) can be derived relating $\ln \Lambda/\mu$ to $1/\varepsilon$ and μ^ε.

The number of renormalization constants is reduced, because ξ is finite from the point of view of renormalization ($\xi_{\mathrm{B}} = \xi$, $Z_\xi = 1$). Instead, the number of independent couplings is not truly reduced, because the constant ξ is still arbitrary. Restrictions have to be imposed on the solution $\eta(\alpha, \xi, \varepsilon)$ to have an effective reduction. Because ξ is finite, therefore a pure number, it is meaningful to investigate criteria that fix the value of ξ unambiguously.

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Leading-log approximation.

Before studying the general solution, it is instructive to work out the solution in the leading-log approximation. The one-loop beta functions have the form

$$\beta_\alpha = \beta_1 \alpha^2, \qquad \beta_\eta = \alpha \left(a + b\eta + c\eta^2\right), \tag{2.12}$$

where β_1, a, b and c are numerical factors. The leading-log solution of equation (2.6) reads

$$\eta_\pm(\alpha, \xi, \varepsilon) = -\frac{1}{2c}\left[b \mp s \frac{1 + \xi\ (-\varepsilon/Z_\alpha)^{\pm s/\beta_1}}{1 - \xi\ (-\varepsilon/Z_\alpha)^{\pm s/\beta_1}}\right], \tag{2.13}$$

where

$$Z_\alpha = \frac{1}{1 - \alpha\beta_1/\varepsilon} \tag{2.14}$$

is the renormalization constant of α and s is the square root of $b^2 - 4ac$. Here, $\eta_\pm$ are just two different ways to write the same solution. If $b^2 - 4ac \geq 0$ we take s to be the positive square root.

For later use, it is convenient to invert the solution (2.13) and write ξ as a function of α and η:

$$\xi = (-\varepsilon/Z_{\alpha_1})^{\mp s/\beta_1} z_\pm, \qquad \text{where} \qquad z_\pm = \frac{b \mp s + 2c\eta_\pm}{b \pm s + 2c\eta_\pm}. \tag{2.15}$$

The bare relations are (2.7). To write them explicitly, we calculate the renormalization constants $\Delta_\eta(\alpha, \eta, \varepsilon)$ of η, using the procedure of the appendix. Basically, define

$$\widetilde{\Delta}_\eta(\alpha, \xi, \varepsilon) = \Delta_\eta\left(\alpha, \eta(\alpha, \xi, \varepsilon), \varepsilon\right),$$

then integrate the equation

$$\frac{\mathrm{d}\left(\alpha\widetilde{\Delta}_\eta\right)}{\mathrm{d}\alpha} = -\frac{\beta_\eta\left(\eta(\alpha, \xi, \varepsilon), \alpha\right)}{\beta_\alpha\left(\eta(\alpha, \xi, \varepsilon), \alpha\right) - \varepsilon\alpha}$$

along the RG flow, with the initial condition $\widetilde{\Delta}_\eta(0, \xi, \varepsilon) < \infty$. Finally insert (2.15) inside $\widetilde{\Delta}_\eta(\alpha, \xi, \varepsilon)$ to eliminate ξ. The result is

$$\Delta_\eta = -\frac{2}{\alpha} \frac{\left(a + b\eta + c\eta^2\right)\left(1 - Z_\alpha^{\pm s/\beta_1}\right)}{b \pm s + 2c\eta - (b \mp s + 2c\eta) Z_\alpha^{\pm s/\beta_1}}. \tag{2.16}$$

Using (2.3), the constant ζ of (2.7) reads

$$\zeta_\pm = -\frac{1}{2c}\left(b \mp s \frac{1 + \xi(-\varepsilon)^{\pm s/\beta_1}}{1 - \xi(-\varepsilon)^{\pm s/\beta_1}}\right). \tag{2.17}$$

Note the similarities between this formula and (2.13). It is useful to invert (2.17) and write also

$$\xi = (-\varepsilon)^{\mp s/\beta_1} z_{\pm\mathrm{B}}, \qquad z_{\pm\mathrm{B}} \equiv \frac{b \mp s + 2c\eta_{\pm\mathrm{B}}}{b \pm s + 2c\eta_{\pm\mathrm{B}}}. \tag{2.18}$$

Observe that if

$$\pm\frac{s}{\beta_1}-1\notin\mathbb{N}, \tag{2.19}$$

the relation (2.17) the function $\zeta(\xi,\varepsilon)$ is not analytic, unless $\xi=0$ or $\xi=\infty$. Then the unique reductions that are analytic both at the renormalized and bare levels are

$$\eta_{\pm}=-\frac{b\mp s}{2c}.$$

They are meaningful only if s is real.

General solution.

Now we study the function $\eta(\alpha,\xi,\varepsilon)$ beyond the leading-log approximation. We first prove that the most general solution $\eta(\alpha,\xi,\varepsilon)$ of (2.6) is analytic in ε at $\alpha\neq 0$. Insert the expansion

$$\eta(\alpha,\xi,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^{i}\eta_{i}(\alpha,\xi) \tag{2.20}$$

into (2.6) and work out the equations for the η_i's. The equation for η_0 is just Zimmermann's equation

$$a+b\eta_0+c\eta_0^2+\sum_{k=2}^{\infty}\alpha^{k-1}Q_{k+1}(\eta_0)=\alpha\eta_0'\left(\beta_1+\sum_{k=2}^{\infty}\alpha^{k-1}P_k(\eta_0)\right). \tag{2.21}$$

Instead, η_i, $i>0$, obey the linear equations

$$\alpha\overline{\beta}\eta_i'=\eta_i\overline{\gamma}+\delta_i\left(\alpha,\eta,\eta'\right), \tag{2.22}$$

where

$$\overline{\gamma}=b+2c\eta_0+\sum_{k=2}^{\infty}\alpha^{k-1}Q_{k+1}'(\eta_0)-\eta_0'\sum_{k=2}^{\infty}\alpha^{k}P_k'(\eta_0),\qquad \overline{\beta}=\beta_1+\sum_{k=2}^{\infty}\alpha^{k-1}P_k(\eta_0),$$

and $\delta_i\left(\alpha,\eta,\eta'\right)$ is analytic in α, polynomial in η_j,η_j' with $j<i$ and does not depend on η_k, η_k' with $k\geq i$.

Next we prove that if

$$\pm\frac{s}{\beta_1}\notin\mathbb{N}, \tag{2.23}$$

then there exists a unique solution $\overline{\eta}(\alpha,\varepsilon)$ that is analytic in α and ε. Indeed, (2.21) has an analytic solution if (2.19) and *a fortiori* (2.23) hold. By induction in i, it is immediate to see that (2.22) admit unique solutions $\eta_i(\alpha)$ that are analytic in α, again if (2.23) hold. Observe that (2.23) is just slightly more restrictive than (2.19), because it excludes also $s=0$.

Finally, we study the most general, non-analytic, solution. The solution of (2.21) has expansions [1]

$$\eta_0(\alpha,\xi)=\eta_{\pm}(\alpha,\xi,0)=\sum_{k=0}^{\infty}\overline{c}_{\pm k}\alpha^{k}+\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\overline{d}_{\pm mn}\xi^{n}\alpha^{m\pm ns/\beta_1}, \tag{2.24}$$

where

$$\overline{c}_{\pm 0} = -\frac{b \mp s}{2c}, \qquad \overline{d}_{\pm 01} = 1,$$

and the other coefficients $\overline{c}_{\pm k}$, $\overline{d}_{\pm mn}$ are unambiguous calculable numbers. The solutions of (2.22) can be worked out inductively in i. Assume that the η_j's, $j < i$, are known. Then

$$\eta_i(\alpha, \xi) = \int_{\overline{\alpha}_i}^{\alpha} \mathrm{d}\alpha' \frac{\delta_i\,(\alpha', \xi)\,\tau(\alpha, \alpha', \xi)}{\alpha' \overline{\beta}(\alpha', \xi)}, \qquad \tau(\alpha, \alpha', \xi) = \exp\left(\int_{\alpha'}^{\alpha} \frac{\overline{\gamma}(\alpha'', \xi)\mathrm{d}\alpha''}{\alpha'' \overline{\beta}(\alpha'', \xi)}\right),$$

where $\overline{\alpha}_i$ is a redundant arbitrary constant, that can be fixed to any non-zero value. Observe that $\eta(\alpha, \xi, \varepsilon)$ is analytic both in ε and ξ at $\alpha \neq 0$, an important property that will be used later.

Comparison of reductions.

The comparison between bare and renormalized reduction relations can be studied generalizing formula (2.24) and using tricks inspired by the ε-expansion. Let

$$\alpha_*(\varepsilon) = \frac{\varepsilon}{\beta_1} + \mathcal{O}(\varepsilon^2), \qquad \eta_{\pm *}(\varepsilon) = -\frac{b \mp s}{2c} + \mathcal{O}(\varepsilon), \tag{2.25}$$

denote the non-trivial RG fixed point at $\varepsilon \neq 0$, namely the solution of

$$\frac{1}{\alpha}\widehat{\beta}_\alpha(\alpha, \eta, \varepsilon) = 0, \qquad \frac{1}{\alpha}\beta_\eta(\alpha, \eta, \varepsilon) = 0.$$

Using (2.5) it is immediate to prove that the solutions are analytic in ε and have the behaviors (2.25), if $\beta_1 \neq 0$ and $s \neq 0$, which we assume. Define the new variables

$$u = \alpha - \alpha_*(\varepsilon), \qquad v = \eta - \eta_{\pm *}(\varepsilon)$$

and write expansions

$$\frac{\widehat{\beta}_\alpha}{\alpha} \equiv f(u, v) = f_1 u + f_2 v + \mathcal{O}(u^2, uv, v^2), \qquad \frac{\beta_\eta}{\alpha} \equiv g(u, v) = g_1 u + g_2 v + \mathcal{O}(u^2, uv, v^2),$$

where $f_1 = \beta_1 + \mathcal{O}(\varepsilon)$, $g_2 = \pm s + \mathcal{O}(\varepsilon)$, $f_2 = \mathcal{O}(\varepsilon^2)$, $g_1 = \mathcal{O}(1)$.

The reduction of couplings is expressed by a function $v(u)$ that satisfies

$$f(u, v(u))\frac{\mathrm{d}v(u)}{\mathrm{d}u} = g(u, v(u)). \tag{2.26}$$

Consider also the equation

$$f(k(v), v) = \frac{\mathrm{d}k(v)}{\mathrm{d}v} g(k(v), v). \tag{2.27}$$

It is simple to see that if

$$\pm\frac{\beta_1}{s} - 1 \notin \mathbb{N} \tag{2.28}$$

then (2.27) admits an analytic solution

$$k(v) = v \sum_{k=0}^{\infty} b_k v^k, \tag{2.29}$$

such that the coefficients b_k are analytic functions of ε and $b_k = \mathcal{O}(\varepsilon^2)$. This ε-behavior can be proved observing that formulas (2.5) imply

$$\left.\frac{\partial^{m+n} f}{\partial u^m \partial v^n}\right|_{u=v=0} = \mathcal{O}(\varepsilon^{n-m}), \qquad \left.\frac{\partial^{m+n+2} g}{\partial u^m \partial v^{n+2}}\right|_{u=v=0} = \mathcal{O}(\varepsilon^{n-m}),$$

if $m \leq n$. Use the solution (2.29) to define $u' = u - k(v)$. Then equations (2.26) can be rewritten in the form

$$\widetilde{f}(u', v(u'))\ u' \frac{\mathrm{d}v(u')}{\mathrm{d}u'} = \widetilde{g}(u', v(u')), \tag{2.30}$$

where $\widetilde{f}$ and $\widetilde{g}$ are analytic functions of u' and v. Precisely $\widetilde{g}(u', v) = g(u, v)$ and

$$\widetilde{f}(u', v) = \frac{1}{u'}\left[f(u' + k(v), v) - \frac{\mathrm{d}k}{\mathrm{d}v} g(u' + k(v), v)\right] = \widetilde{f}_1 + \mathcal{O}(u', v),$$

and $\widetilde{f}_1 = \beta_1 + \mathcal{O}(\varepsilon)$. Finally, it is easy to see that the solution of (2.30) has expansions

$$v_\pm(u', \xi) = \sum_{k=1}^{\infty} c'_{\pm k}(u')^k + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} d'_{\pm mn} \xi^n (u')^{m+nQ_\pm}, \qquad Q_\pm = \pm\frac{s}{\beta_1} + \mathcal{O}(\varepsilon), \tag{2.31}$$

where the coefficients $c'_{\pm k}$, $d'_{\pm mn}$ are unambiguous calculable functions of ε ($d'_{\pm 01}$ being set to 1), and ξ is the arbitrary constant. If $s/\beta_1 > 0$ the meaningful expansions are $v_+(u', \xi)$ and $v_-(u', 0)$, if $s/\beta_1 < 0$ they are $v_+(u', 0)$ and $v_-(u', \xi)$.

The invertibility conditions (2.28), necessary to write the solution $k(v)$ and (2.31), are not included in (2.23). Typically, s is irrational or complex, so it is not difficult to fulfill both (2.23) and (2.28).

Using (2.31) and the variable changes performed so far, it is possible to express the solution (2.31) in the α-η parametrization, and write $\eta(\alpha, \xi, \varepsilon)$. It is then easy to show again that $\eta(\alpha, \xi, \varepsilon)$ is analytic both in ε and ξ at $\alpha \neq 0$.

Relation between the renormalized and bare arbitrary constants ξ and ζ.

Inserting (2.31) into (2.9) the $\alpha \to 0$ limit gives an implicit equation for $\zeta_\pm$, that can be solved recursively in powers of ε and $\xi\varepsilon^{Q_\pm}$. Observe that

$$\lim_{\alpha \to 0} u' = -\alpha_*(\varepsilon) - k\left(\zeta_\pm - \eta_{\pm *}(\varepsilon)\right) = \varepsilon\left(-\frac{1}{\beta_1} + \mathcal{O}(\varepsilon, \varepsilon\zeta_\pm, \ldots)\right).$$

The relation $\zeta_\pm(\varepsilon, \xi)$ is non-analytic in ε at $\xi \neq 0$, so the double-analyticity requirement correctly implies $\xi = 0$ and selects the reduction uniquely.

In the leading-log approximation the known results can be immediately recovered.

Relation between the two criteria for the reduction.

The limit (2.9) not only explains the similarities between $\eta(\alpha, \xi, \varepsilon)$ and $\zeta(\varepsilon, \xi)$, exhibited by (2.13) and (2.17) in the leading-log approximation, but also establishes a connection between the two criteria for the reduction, namely analyticity in α and ε at the origin, and analyticity in ε at the bare and renormalized levels. The interpolation between the two requirements is encoded in the function $u' = \alpha - \varepsilon/\beta_1 + \cdots$. Raised to non-integer powers, u' originates both the non-integer powers of α in the $\varepsilon \to 0$ limit and the non-integer powers of ε in the $\alpha \to 0$ limit. The "dual" roles played by α and ε are not surprising, if we recall that in the perturbative regime

$$\ln \Lambda \sim \frac{1}{\varepsilon}, \qquad \alpha \sim \frac{1}{\ln \Lambda},$$

where Λ denotes a cut-off, so $\alpha \sim \varepsilon$.

3. Infinite reduction

In this section we study the infinite reduction for non-renormalizable theories at the regularized level, using the dimensional-regularization technique. The minimal subtraction scheme is used for the unreduced theory. The subtraction scheme of the reduced theory is the one induced by the reduction itself. Unless otherwise specified, the words "relevant", "marginal" and "irrelevant" refer to the gaussian fixed point, so they are equivalent to "super-renormalizable", "strictly renormalizable" and "non-renormalizable", respectively. In the study of deformations of interacting conformal field theories, our construction allows also to characterize the deformation as marginal, relevant or irrelevant at the interacting fixed point.

For definiteness, we work in four dimensions. The generalization to odd and other even dimensions is direct and left to the reader. Let $\mathcal{R}$ denote the power-counting renormalizable subsector of the theory. For the moment we assume that $\mathcal{R}$ contains a single marginal coupling $\alpha = g^2$, where α multiplies the four-leg marginal vertices and g multiplies the three-leg marginal vertices. The power-counting renormalizable sector $\mathcal{R}$ needs to be fully interacting, because the infinite reduction does not work when the marginal sector is free or only partially interacting. We assume also that $\mathcal{R}$ does not contain relevant couplings. This assumption ensures that the beta functions of the irrelevant sector depend polynomially on the irrelevant couplings, so there exists a simple recursive procedure to solve the reduction equations. When $\mathcal{R}$ is fully interacting, relevant parameters can be added *after* the construction of the irrelevant deformation and studied perturbatively in the reduction equations. In practice, interactions have to be turned on in the following order: first the marginal interactions, then the irrelevant interactions, finally the relevant interactions.

Let $\mathcal{O}_n$ denote a basis of local, "essential", scalar, symmetric, canonically irrelevant operators of $\mathcal{R}$. Essential operators are defined as the equivalence classes of operators that differ by total derivatives, terms proportional to the field equations and BRST-exact terms [7]. Total derivatives are trivial in perturbation theory. The terms proportional to the field equations can be renormalized away by means of field redefinitions. Finally,

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the BRST-exact sector does not affect the physical quantities. "Symmetric" means that the integrated operators have to be invariant under the non-anomalous symmetries of the theory.

The irrelevant terms can be ordered according to their "level". If, at $\varepsilon = 0$, the operator $\mathcal{O}_n$ has canonical dimensionality d_n in units of mass, then the level n of $\mathcal{O}_n$ is the difference $d_n - D$, D being the physical spacetime dimension. In general, each level contains finitely many operators $\mathcal{O}_n^I$, which can mix under renormalization.

It is convenient to write the classical lagrangian in the form

$$\mathcal{L}_{\rm cl}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, g] + \sum_{n>0} \sum_I g^{2p_n^I} \lambda_n^I \mathcal{O}_n^I(\varphi), \tag{3.1}$$

where φ generically denotes the fields of the theory, $\mathcal{L}_{\mathcal{R}}[\varphi, g]$ is the lagrangian of the renormalizable subsector and $p_n^I = N_n^I/2 - 1$, where N_n^I is the number of legs of $\mathcal{O}_n^I$. The bare couplings are

$$\alpha_{\rm B} = g_{\rm B}^2 = \alpha\mu^\varepsilon Z_\alpha(\alpha, \varepsilon), \qquad \lambda_{n{\rm B}}^I = \lambda_n^J Z_n^{IJ}(\alpha, \lambda, \varepsilon), \tag{3.2}$$

and the bare lagrangian reads

$$\mathcal{L}_{\rm cl\ B}[\varphi_{\rm B}] = \mathcal{L}_{\mathcal{R}\ {\rm B}}[\varphi_{\rm B}, g_{\rm B}] + \sum_{n>0} \sum_I \lambda_{n{\rm B}}^I g_{\rm B}^{2p_n^I} \mathcal{O}_n^I(\varphi_{\rm B}).$$

Call *dimensionality-defect* p of a quantity the difference between its dimensionality at $\varepsilon \neq 0$ and its dimensionality at $\varepsilon = 0$, divided by ε. For example, the dimensionality-defect of $\alpha_{\rm B}$ is 1 and the one of $g_{\rm B}$ is 1/2. Assume that the higher-derivative kinetic terms of the fields are conventionally multiplied by unity. Then a *minimal* coupling χ is the coefficient of a vertex. In symbolic notation, the vertex reads

$$\chi[\partial^q]\phi^{n_s}\psi^{n_f}A^{n_v}G^{n_g},$$

where $[\partial^q]$ stands for q variously distributed derivatives, n_s, n_f, n_v and n_g are the numbers of scalar, fermion, vector and graviton legs, respectively. The dimensionality-defect of $\chi_{\rm B}$ is

$$p_\chi = \frac{N}{2} - 1, \qquad N = n_s + n_f + n_v + n_g, \tag{3.3}$$

where N is the total number of legs. Thus, p_χ is greater than zero whenever $N > 2$. One-leg terms are associated with scalar vacuum expectation values and the cosmological term. The two-leg terms include mass terms, kinetic terms and contributions of the cosmological term. In very general situations, including gravity [16], the higher-derivative kinetic terms can be converted into vertices, mass terms and the cosmological term, using the field equations. As mentioned above, in the construction of irrelevant deformations by means of the infinite reduction, the relevant parameters are initially turned off (they can be turned on perturbatively at a secondary stage). Therefore, for the purposes of the infinite reduction, we can assume that there are no independent quadratic terms besides the free kinetic ones and that the vertices have three legs or more, which ensures that every essential

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coupling has $p > 0$. The factors $g^{2p_n^I}$ of formula (3.1) have been introduced so that the dimensionality-defects of the non-minimal irrelevant couplings $\lambda_{n\ell\mathrm{B}}^I$ are zero. Observe that, by definition, also the marginal vertices contained in $\mathcal{R}$ are multiplied by g-powers equal to the number of legs minus two.

It is useful to define a parity transformation U, that sends g into $-g$ and every field φ into $-\varphi$. Clearly, $\mathcal{R}$ is U-invariant. Then it is evident that (3.1) is U-invariant and each λ_n^I is U-invariant. The parametrization (3.1) is convenient because it allows us to work with U-even quantities.

Using the minimal subtraction scheme, the RG equations read

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu} = \widehat{\beta}_\alpha(\alpha,\varepsilon) = \beta_\alpha(\alpha) - \varepsilon\alpha, \qquad \frac{\mathrm{d}\lambda_n^I}{\mathrm{d}\ln\mu} = \beta_n^I(\alpha,\lambda). \tag{3.4}$$

The beta function of the irrelevant couplings λ_n have the form

$$\beta_n^I(\lambda,\alpha) = \gamma_n^{IJ}(\alpha)\ \lambda_n^J + \alpha\delta_n^I(\alpha,\lambda), \tag{3.5}$$

where $\delta_n^I(\alpha,\lambda)$ depends polynomially, at least quadratically, on the irrelevant couplings λ_m with $m < n$ and does not depend on the irrelevant couplings with $m \geq n$. Instead, $\gamma_n^{IJ}(\alpha)$ is the matrix of anomalous dimensions of the operators $g^{2p_n^I}\mathcal{O}_n^I(\varphi)$ (defined up to total derivatives and terms proportional to the field equations), calculated in the undeformed theory $\mathcal{R}$. Both $\delta_n^I(\alpha,\lambda)$ and $\gamma_n^{IJ}(\alpha)$ are analytic in α and $\gamma_n^{IJ}(\alpha)$ is of order α.

The structure (3.5) follows from dimensional analysis and simple diagrammatics. In perturbation theory only non-negative powers of the couplings can appear and by assumption the theory does not contain parameters with positive dimensionalities in units of mass. Then, matching the dimensionalities of the left- and right-hand sides of (3.5), the λ-dependence of the right-hand side of (3.5) follows. As far as the α-dependence is concerned, let G be a diagram contributing to the renormalization of the vertex $\mathcal{O}_n^J(\varphi)$, with $E = N_n^J$ external legs, I internal legs and V vertices. The g-powers carried by the (marginal and irrelevant) vertices due to the fully non-minimal parametrization (3.1) are equal to

$$\sum_{\text{vertices}} (\#\text{legs} - 2) = E + 2I - 2V, \tag{3.6}$$

recalling that the total number of legs attached to the vertices is $E + 2I$. Since $I - V = L - 1 \geq 0$, where L is the number of loops, the total g-power carried by the diagram is $N_n^J - 2 + 2L = 2p_n^J + 2L$. Thus

$$\mu^{-p_n^J\varepsilon} g_{\mathrm{B}}^{2p_n^J} \lambda_{n\mathrm{B}}^J = g^{2p_n^J}\lambda_n^J + \sum_{L\geq 1} g^{2p_n^J}\alpha^L d_{n,L}^J(\varepsilon,\lambda), \tag{3.7}$$

where $d_{n,L}^J(\varepsilon,\lambda)$ are divergent coefficients that do not depend on the λ_k's with $k > n$. Since $Z_g = Z_\alpha^{1/2} = 1 + \sum_{L\geq 1}\alpha^L c_L(\varepsilon)$, (3.7) gives immediately

$$\lambda_{n\mathrm{B}}^J = \lambda_n^J + \sum_{L\geq 1}\alpha^L d'^{\,J}_{n,L}(\varepsilon,\lambda), \qquad \lambda_n^J = \lambda_{n\mathrm{B}}^J + \sum_{L\geq 1}\mu^{-\varepsilon L}\alpha_{\mathrm{B}}^L d''^{\,J}_{n,L}(\varepsilon,\lambda_{\mathrm{B}}), \tag{3.8}$$

for other divergent coefficients $d'^{\,J}_{n,L}$, $d''^{\,J}_{n,L}$ that do not depend on the λ_k's with $k > n$. Thus the beta function of λ_n^J has the α-dependence specified in (3.5).

Another way to derive (3.5) is as follows. Define rescaled fields $\varphi = \varphi'/g$. Then each primed field is U-even and (3.1) becomes

$$\mathcal{L}'_{\mathrm{cl}}[\varphi'] = \frac{1}{\alpha}\left[\mathcal{L}'_{\mathcal{R}}[\varphi'] + \sum_n \sum_J \lambda_n^J \mathcal{O}_n^J(\varphi')\right]. \tag{3.9}$$

In this parametrization, every propagator carries a factor α and every vertex carries a factor $1/\alpha$. So, a diagram with I internal legs, L loops and V vertices carries a factor $\alpha^{I-V} = \alpha^L/\alpha$, that gives a counterterm contributing to $\lambda_{n\mathrm{B}}^J/\alpha_{\mathrm{B}}$. Thus (3.8) follow immediately and the beta function of λ_n^J is a sum of contributions proportional to α^L, $L \geq 1$, in agreement with (3.5).

Observe that the structures (3.1) and (3.9) are compatible both with gauge invariance and the use of field equations.

The α-beta function and the anomalous dimensions of $\mathcal{R}$ have expansions

$$\beta_\alpha = \alpha^2 \beta_\alpha^{(1)} + \mathcal{O}(\alpha^3), \qquad \gamma_n(\alpha) = \alpha\gamma_n^{(1)} + \mathcal{O}(\alpha^2), \tag{3.10}$$

etc. and we assume $\beta_\alpha^{(1)} \neq 0$. For the moment it is convenient to ignore the indices labelling operators of the same level, which amounts to discard the renormalization mixing.

Now we study the infinite reduction. As usual, the reduced deformation is made of a head and a queue. The head is the irrelevant term of lowest dimensionality, whose level is denoted with ℓ . The queue is made of terms of levels $n\ell$, with n integer. The head is multiplied by the irrelevant coupling λ_ℓ, while the terms of the queue are multiplied by functions of λ_ℓ and the marginal couplings α of $\mathcal{R}$. Write

$$\mathcal{L}_{\mathrm{cl}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, g] + g^{2p_\ell}\lambda_\ell \mathcal{O}_\ell(\varphi) + \sum_{n>1} g^{2p_{n\ell}} \lambda_{n\ell}(\alpha, \lambda_\ell, \varepsilon)\ \mathcal{O}_{n\ell}(\varphi). \tag{3.11}$$

Renormalized reduction relations.

On dimensional grounds, the renormalized reduction relations have the form

$$\lambda_{n\ell}(\alpha, \lambda_\ell, \varepsilon) = \lambda_\ell^n f_n(\alpha, \varepsilon), \qquad n > 1. \tag{3.12}$$

The lowest level of the deformation has $\delta_\ell = 0$, so

$$\beta_\ell(\lambda, \alpha, \varepsilon) = \lambda_\ell\ \gamma_\ell(\alpha). \tag{3.13}$$

Assume inductively that the functions $f_m(\alpha, \xi, \varepsilon)$ are known for $m < n$ and that they depend on certain constants ξ_m, $m < n$. Then, since $\delta_n(\alpha, \lambda)$ depends only on the irrelevant couplings $\lambda_{m\ell}$ with $m < n$, it is possible to write

$$\delta_{n\ell}(\alpha, \lambda, \varepsilon) = \overline{\delta}_n(\alpha, \xi, \varepsilon)\ \lambda_\ell^n, \tag{3.14}$$

where $\overline{\delta}_n(\alpha, \xi, \varepsilon)$ are known functions that depend on ξ_k with $k < n$. Differentiating (3.12) and using (3.5) and (3.13) we obtain the equation

$$f_n'(\alpha, \varepsilon)\ \widehat{\beta}_\alpha = f_n(\alpha, \varepsilon)\ \left(\gamma_{n\ell}(\alpha) - n\gamma_\ell(\alpha)\right) + \alpha\overline{\delta}_n(\alpha, \xi, \varepsilon). \tag{3.15}$$

These are the RG consistency conditions for the functions $f_n(\alpha,\varepsilon)$. They are first order differential equations, so the solutions contain arbitrary constants ξ_n, one for every n. Since the beta function of $\lambda_{n\ell}$ depends only on the $\lambda_{k\ell}$'s with $k \leq n$, the function $f_n(\alpha,\xi,\varepsilon)$ depends only the constants ξ_k with $k \leq n$. The solution of (3.15) can be split into a sum of two terms,

$$f_n(\alpha,\xi,\varepsilon) = \overline{f}_n(\alpha,\xi,\varepsilon) + \xi_n \overline{s}_n(\alpha,\varepsilon), \tag{3.16}$$

where $\overline{s}_n(\alpha,\varepsilon)$ solves the homogeneous equation, ξ_n is the arbitrary integration constant belonging to f_n and $\overline{f}_n(\alpha,\xi,\varepsilon)$ is a particular solution, that depends on the constants ξ_k with $k < n$.

If $\overline{\delta}_n \neq 0$ the solution can be written also as

$$f_n(\alpha,\xi,\varepsilon) = \int_{\xi_n'}^{\alpha} \mathrm{d}\alpha' \frac{\alpha' \overline{\delta}_n(\alpha',\xi,\varepsilon)\ s_n(\alpha,\alpha',\varepsilon)}{\widehat{\beta}_\alpha(\alpha',\varepsilon)}, \tag{3.17}$$

where

$$s_n(\alpha,\alpha',\varepsilon) = \exp\left(\int_{\alpha'}^{\alpha} \mathrm{d}\alpha'' \frac{\gamma_{n\ell}(\alpha'') - n\gamma_\ell(\alpha'')}{\widehat{\beta}_\alpha(\alpha'',\varepsilon)}\right) \tag{3.18}$$

and the ξ_n''s are constants suitably related to the ξ_n's.

If $\beta_\alpha \neq 0$ the solutions are analytic in ε. Indeed, assume by induction that $f_m(\alpha,\xi,\varepsilon)$, $m < n$, are analytic in ε. Then $\overline{\delta}_n(\alpha,\xi,\varepsilon)$ is analytic in ε and (3.17)-(3.18) show that also $f_n(\alpha,\xi,\varepsilon)$ is analytic in ε.

At $\varepsilon \neq 0$ the solutions are also analytic in α. Indeed, for α small $\widehat{\beta}_\alpha(\alpha',\varepsilon) \sim -\alpha'\varepsilon$. Assume by induction that $f_m(\alpha,\xi,\varepsilon)$, $m < n$, are analytic in α. Then $\overline{\delta}_n(\alpha,\xi,\varepsilon)$ is analytic in α and (3.17)-(3.18) show that also $f_n(\alpha,\xi,\varepsilon)$ is analytic in α.

From the point of view of renormalization, the constants ξ are finite arbitrary parameters ($Z_\xi = 1$) and the divergences of (3.11) are removed by means of renormalization constants for α and λ_ℓ, plus field redefinitions, with no independent renormalization constants for the couplings $\lambda_{n\ell}$, $n > 1$, of the queue. This fact can be proved with an argument analogous to the one leading to (2.11) (see also [3]).

Now we prove that if certain invertibility conditions are fulfilled, then there exists a unique solution $\overline{f}_n(\alpha,\varepsilon)$ that is analytic both in α and ε.

Doubly analytic solution.

If the invertibility conditions

$$\tau_k \equiv \frac{\gamma_{k\ell}^{(1)} - k\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}} \notin \mathbb{N}, \qquad k > 1 \tag{3.19}$$

hold, then there exists a unique solution $\overline{f}_k(\alpha,\varepsilon)$ that is analytic in α and ε. Use the ε-analyticity of $f_n(\alpha,\varepsilon)$ at $\alpha \neq 0$ to write the expansion

$$\overline{f}_k(\alpha,\varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \overline{f}_{i,k}(\alpha). \tag{3.20}$$

Assume inductively that (3.20) are analytic both in α and ε for $k < n$. Then the functions $\overline{f}_{i,k}(\alpha)$, $k < n$, are analytic in α and we can write

$$\overline{\delta}_n(\alpha, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \overline{\delta}_{i,n}(\alpha), \tag{3.21}$$

where $\overline{\delta}_{i,n}(\alpha)$ are analytic in α. Insert (3.20) and (3.21) into (3.15) and write the reduction equations as

$$\beta_\alpha \frac{\mathrm{d}\overline{f}_{i,n}(\alpha)}{\mathrm{d}\alpha} = \overline{f}_{i,n}(\alpha) \;\; (\gamma_{n\ell}(\alpha) - n\gamma_\ell(\alpha)) + \alpha \frac{\mathrm{d}\overline{f}_{i-1,n}(\alpha)}{\mathrm{d}\alpha} + \alpha \overline{\delta}_{i,n}(\alpha), \tag{3.22}$$

$i = 0, 1, \ldots$, with $\overline{f}_{-1,n}(\alpha) = 0$. The solution of (3.22) can be worked out recursively in i and, for given i, in power series of α. It is then immediate to see that if the invertibility conditions (3.19) hold, there exist unique analytic solutions $\overline{f}_{i,n}(\alpha)$.

The arbitrary constant ξ_n multiplies the function $\overline{s}_n(\alpha, \varepsilon)$, which is not analytic in both α and ε, when the invertibility conditions are fulfilled. Indeed, $\overline{s}_n(\alpha, \varepsilon)$ is proportional to $s_n(\alpha, \overline{\alpha}, \varepsilon)$, with $\overline{\alpha} > 0$. Expanding $s_n(\alpha, \overline{\alpha}, \varepsilon)$ in powers of ε it is immediate to see that the terms of the expansion are not analytic in α around $\alpha \sim 0$. Instead, at $\varepsilon \neq 0$ $s_n(\alpha, \overline{\alpha}, \varepsilon)$ is obviously analytic in α, since it is just a product of renormalization constants of the undeformed theory $\mathcal{R}$.

Violations of the invertibility conditions.

When some invertibility conditions are violated, namely $\tau_{\overline{n}} = \overline{r} \in \mathbb{N}$ for some $\overline{n}$, then it is necessary to introduce a new independent coupling. Consider (3.22). The solution $\overline{f}_{0,\overline{n}}(\alpha)$ can be worked out in power series of α up to the order $\alpha^{\overline{r}-1}$, while the coefficient of $\alpha^{\overline{r}}$ is ill-defined: the problem is avoided introducing a new independent coupling $\lambda_{\overline{n}\ell}^{(0)}$ at order $\alpha^{\overline{r}}$. Similarly, the solutions $\overline{f}_{i,\overline{n}}(\alpha)$, $0 < i < \overline{r}$, can be worked out up to the orders $\alpha^{\overline{r}-i-1}$, but they require new couplings $\lambda_{\overline{n}\ell}^{(i)}$ from the orders $\alpha^{\overline{r}-i}$ on. However, only $\lambda_{\overline{n}\ell}^{(0)}$ is a new physical coupling, because the $\lambda_{\overline{n}\ell}^{(i)}$'s with $i > 0$ belong to the evanescent sector, so they do not affect the physical quantities. These properties ensure that the violations of the invertibility conditions are less harmful than they appear at first sight: certainly they cause the introduction of new parameters, possibly infinitely many, but in general τ_n grows with n and the physical new couplings appear at higher and higher orders, thus permitting low-order predictions with a relatively small number of independent couplings [3].

It is convenient to introduce $\overline{r} + 1$ new parameters $\lambda_{\overline{n}\ell}^{(i)}$, $i = 0, 1, \ldots \overline{r}$, and write

$$\lambda_{\overline{n}\ell} = f_{\overline{n}}(\alpha, \varepsilon) \lambda_\ell^{\overline{n}} + \sum_{i=0}^{\overline{r}} \alpha^{\overline{r}-i} \varepsilon^i \lambda_{\overline{n}\ell}^{(i)}, \qquad \widehat{\beta}_{\overline{n}_\ell}^{(i)} = \gamma_{\overline{n}\ell}^{(i)} \lambda_{\overline{n}\ell}^{(i)} + \varepsilon(\overline{r} - i) \lambda_{\overline{n}\ell}^{(i)} + \alpha \lambda_\ell^{\overline{n}} \delta_{\overline{n}\ell}^{(i)}(\alpha, \varepsilon), \tag{3.23}$$

where $f_{\overline{n}}(\alpha, \varepsilon)$ is determined up to the orders $\alpha^{\overline{r}-1-i}\varepsilon^i$, $i = 0, \ldots \overline{r}-1$, solving the reduction equations (3.22), $\gamma_{\overline{n}\ell}^{(i)} = \gamma_{\overline{n}\ell} - (\overline{r} - i)\beta_\alpha/\alpha = \mathcal{O}(\alpha)$ and $\delta_{\overline{n}\ell}^{(i)}$ are analytic in α. As anticipated above, the new physical parameter $\lambda_{\overline{n}\ell}^{(0)}$ is multiplied by $\alpha^{\overline{r}}$. For $n > \overline{n}$ write

$$\lambda_{n\ell} = \sum_{\{m\}} f_{n,\{m\}}(\alpha, \varepsilon) \; \lambda_\ell^{\widehat{m}} \prod_{i=0}^{\overline{r}} \left(\alpha^{\overline{r}-i} \varepsilon^i \lambda_{\overline{n}\ell}^{(i)} \right)^{m_i}, \tag{3.24}$$

where $\widehat{m}, m_i$ are integers such that $\widehat{m} + \overline{n}\sum_{i=0}^{\overline{r}} m_i = n$. In (3.5) $\delta_n(\alpha, \lambda)$ can be decomposed as

$$\delta_n(\alpha, \lambda) = \sum_{\{m\}} \delta_{n,\{m\}}(\alpha, \varepsilon)\ \lambda_\ell^{\widehat{m}} \prod_{i=0}^{\overline{r}} \left(\alpha^{\overline{r}-i}\varepsilon^i \lambda_{\overline{n}\ell}^{(i)}\right)^{m_i}.$$

Then (3.5) give equations of the form

$$\widehat{\beta}_\alpha f'_{n,\{m\}} = \left(\gamma_{n\ell} - \widehat{m}\gamma_\ell - \sum_{j=0}^{\overline{r}} m_j \gamma_{\overline{n}\ell}\right) f_{n,\{m\}} + \alpha \widehat{\delta}_{n,\{m\}}(\alpha, f, \varepsilon) \tag{3.25}$$

where $\widehat{\delta}_{n,\{m\}}(\alpha, f, \varepsilon)$ depends on the functions $f_{k,\{m'\}}$ with $k < n$ and $f_{n,\{m'\}}$ with $\widehat{m}' < \widehat{m}$. The invertibility conditions are still (3.19) for $n > \overline{n}$, because the one-loop coefficient of the combination of anomalous dimensions written in the parenthesis of (3.25) is equal to

$$\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)} - \beta_\alpha^{(1)} \overline{r} \sum_{j=0}^{\overline{r}} m_j.$$

Then, it is possible to solve (3.25) recursively in $\widehat{m}$ for given n and there exist unique solutions $f_{n,\{m\}}(\alpha, \varepsilon)$ that are analytic in α and ε.

We remark that the appearance of new parameters is guided by the construction itself. Moreover, the invertibility conditions (3.19) depend only on the one-loop $\mathcal{R}$ beta function and the one-loop anomalous dimensions of the composite operators of $\mathcal{R}$. Such quantities are calculable in the undeformed renormalizable subsector $\mathcal{R}$, before turning the irrelevant deformation on. Thus, it is possible to count the parameters of the non-renormalizable interaction, or say how dense they are, before constructing the non-renormalizable interaction. This property emphasized once again the perturbative character of the infinite reduction.

Renormalization mixing.

Taking into account of the renormalization mixing, the analysis generalizes in a simple way. Assume for the moment that the head does not mix. Distinguish the couplings of the same level with extra indices $I, J, \ldots$ Write $\lambda_{n\ell}^I(\alpha, \lambda_\ell, \varepsilon) = \lambda_\ell^n f_n^I(\alpha, \varepsilon)$, $n > 1$, $\overline{f}_n^I(\alpha, \varepsilon) = \sum_{i=0}^\infty \varepsilon^i \overline{f}_{i,n}^I(\alpha)$. Formula (3.15) becomes

$$\beta_\alpha \frac{\mathrm{d}\overline{f}_{i,n}^I(\alpha)}{\mathrm{d}\alpha} = \left(\gamma_{n\ell}^{IJ}(\alpha) - n\delta^{IJ}\gamma_\ell(\alpha)\right) \overline{f}_{i,n}^J(\alpha) + \alpha \frac{\mathrm{d}\overline{f}_{i-1,n}^I(\alpha)}{\mathrm{d}\alpha} + \alpha \overline{\delta}_{i,n}^I(\alpha), \tag{3.26}$$

which admit unique analytic solutions $\overline{f}_{i,n}^I(\alpha)$ if the matrices

$$\frac{\gamma_{n\ell}^{(1)IJ} - n\delta^{IJ}\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}, \tag{3.27}$$

$n > 1$, have no integer eigenvalue.

If the head itself has a non-trivial mixing with other operators, call $\gamma_\ell^{(1)}$ a real eigenvalue of $\gamma_\ell^{(1)IJ}$. We assume for simplicity that the eigenvalue $\gamma_\ell^{(1)}$ has multiplicity one. Perform a constant redefinition $\lambda_\ell^I \to M^{IJ}\lambda_\ell^J$, M^{IJ}=constant, to put the matrix $\gamma_\ell^{(1)IJ}$ into its canonical Jordan form with $\gamma_\ell^{(1)NN} = \gamma_\ell^{(1)}$, $\gamma_\ell^{(1)N\overline{J}} = \gamma_\ell^{(1)\overline{I}N} = 0$. Here the unoverlined indices I, J range from 1 to N and the overlined indices $\overline{I}$, $\overline{J}$ range from 1 to $N-1$. Then take $\lambda_\ell \equiv \lambda_\ell^N$ as independent coupling and reduce the other level-ℓ couplings as

$$\lambda_\ell^{\overline{I}} = f^{\overline{I}}(\alpha,\varepsilon)\lambda_\ell.$$

The level-ℓ beta functions $\beta_\ell^I = \gamma_\ell^{IJ}\lambda_\ell^J$ give

$$\beta_\ell = \beta_\ell^N = \left(\gamma_\ell^{NN} + \gamma_\ell^{N\overline{I}} f^{\overline{I}}\right)\lambda_\ell, \qquad \widehat{\beta}_\alpha \frac{\mathrm{d}f^{\overline{I}}}{\mathrm{d}\alpha} = \left(\gamma_\ell^{\overline{I}\overline{J}} - \delta^{\overline{I}\overline{J}}\gamma_\ell^{NN}\right) f^{\overline{J}} + \gamma_\ell^{\overline{I}N} - f^{\overline{I}}\gamma_\ell^{N\overline{J}} f^{\overline{J}}. \tag{3.28}$$

The second equation depends quadratically on the $f^{\overline{I}}$'s, but the quadratic term is multiplied by $\gamma_\ell^{N\overline{J}}$, which is $\mathcal{O}(\alpha^2)$ by construction. Writing $f^{\overline{I}}(\alpha,\varepsilon) = \sum_{i=0}^{\infty}\varepsilon^i f_i^{\overline{I}}(\alpha)$ as usual, the equations for $f_i^{\overline{I}}(\alpha)$ can be solved recursively in i and in power series of α. The doubly analytic solution exists and is unique if the matrices

$$\frac{\gamma_\ell^{(1)\overline{I}\overline{J}} - \delta^{\overline{I}\overline{J}}\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}}$$

have no integer eigenvalue. The functions $f^{\overline{I}}$ determine also the beta function β_ℓ. Observe that the anomalous dimension $\gamma_\ell \equiv \gamma_\ell^{NN} + \gamma_\ell^{N\overline{I}} f^{\overline{I}}$ of the coupling λ_ℓ is equal to $\alpha\gamma_\ell^{(1)} + \mathcal{O}(\alpha^2)$. At higher levels the reduction proceeds as usual and the invertibility conditions are still that the matrices (3.27) have no integer eigenvalue for $n > 1$. The head of the deformation is $\sum_I g^{2p_\ell^I}\lambda_\ell^I\mathcal{O}_\ell^I(\varphi)$. If the eigenvalue $\gamma_\ell^{(1)}$ is complex it is necessary to consider a two-head deformation involving also its complex conjugate [3].

Perturbative meromorphy.

The reduced theory reads

$$\mathcal{L}_{\mathrm{cl}}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, g] + \sum_{n=1}^{\infty}\sum_I g^{2p_{n\ell}^I}\lambda_\ell^n \overline{f}_n^I(\alpha,\varepsilon)\ \mathcal{O}_{n\ell}^I(\varphi). \tag{3.29}$$

Since $p_{n\ell}^I > 0$ every term of the irrelevant deformation is parametrized in a non-minimal way and in the $g \to 0$ limit at fixed λ_ℓ the theory becomes free. In this parametrization, $\lambda_\ell = 1/M_{P\mathrm{eff}}^\ell$ defines the *effective Planck mass* $M_{P\mathrm{eff}}$ and the perturbative expansion in powers of the energy E is meaningful for $E \ll M_{P\mathrm{eff}}$. On the other hand, define the *Planck mass* $M_P = g^{-2\overline{p}/\ell}\lambda_\ell^{-1/\ell}$, in such a way that the irrelevant terms with dimensionality-defect $\overline{p}$ are coupled in a minimal way. Then we get

$$\mathcal{L}[\varphi] = \mathcal{L}_{\mathcal{R}}[\varphi, g] + \sum_{n=1}^{\infty}\sum_I g^{2\widetilde{p}_{n\ell}^I}\overline{f}_n^I(\alpha,\varepsilon)\ M_P^{-n\ell}\ \mathcal{O}_{n\ell}^I(\varphi), \tag{3.30}$$

where $\widetilde{p}_{n\ell}^I = p_{n\ell}^I - n\overline{p}$. Most of the numbers $\widetilde{p}_{n\ell}^I$ are negative, so the $g \to 0$ limit at fixed M_P is singular. Nevertheless, the singularity is bounded by the order of the perturbative expansion and indeed can be reabsorbed into the effective Planck mass. Because of this feature, the reduction is said to be *perturbatively meromorphic* [3]. Since the g-singularities of (3.30) can be reabsorbed only in a fully non-minimal parametrization, there is no way to turn the marginal interaction off, keeping the irrelevant interaction on, which is why the renormalizable sector $\mathcal{R}$ needs to be fully interacting.

4. Bare infinite reduction

In this section we study the infinite reduction at the bare level and relate it to the renormalized infinite reduction studied in the previous section. We want to derive an alternative criterion to select the infinite reduction. In the previous section we showed that the infinite reduction follows from double analyticity in α and ε of the renormalized reduction relations. Here we do not pay attention to the α-dependence and show that $\overline{f}_k(\alpha, \varepsilon)$ is also the unique solution such that both the bare and renormalized reduction relations are analytic in ε.

Bare reduction relations.

At the level of bare couplings, the reduction has a simpler form, namely

$$\lambda_{n\ell\mathrm{B}} = \zeta_n \lambda_{\ell\mathrm{B}}^n, \tag{4.1}$$

where the ζ_n's are constants. This expression is fixed matching the naive dimensionalities and the dimensionality-defects and demanding analyticity in $\lambda_{\ell\mathrm{B}}$ and α_{B}. Since $\lambda_{n\ell\mathrm{B}}$ and $\lambda_{\ell\mathrm{B}}$ are dimensionless at $\varepsilon \neq 0$ and α_{B} is dimensionful, the bare relations do not depend on α_{B}.

Now, rewrite (4.1) in terms of the renormalized couplings, using (3.2):

$$\lambda_{n\ell} Z_{n\ell}(\alpha, \lambda, \varepsilon) = \zeta_n \lambda_\ell^n Z_\ell^n(\alpha, \varepsilon). \tag{4.2}$$

These are algebraic equations relating $\lambda_{n\ell}$ with λ_ℓ and α. In particular, they can be used to work out another expression of $f_n(\alpha, \xi, \varepsilon)$, that must be equivalent to (3.16) and (3.17) once ζ and ξ are suitably related.

The main goal of this section is to prove that if the invertibility conditions (3.19) are fulfilled, there exists unique values of ξ_n and ζ_n such that both (3.16) and (4.1) are analytic in ε. In our notation these values are $\xi_n = 0$ for every n, while ζ_n are equal to suitable analytic functions $\overline{\zeta}_n(\varepsilon)$ of ε.

Equivalence of the bare and renormalized reduction equations.

First we prove that the algebraic relations (4.2) are equivalent to the differential equations (3.15), once ξ_n and ζ_n are appropriately related to each other.

Coherently with (3.5), it is convenient to decompose $\lambda_{n\ell} Z_{n\ell}$ into a sum of two contributions: the terms that are linear in $\lambda_{n\ell}$ plus the terms that are at least quadratic in the irrelevant couplings with lower levels. Precisely,

$$\lambda_{n\ell} Z_{n\ell}(\alpha, \lambda, \varepsilon) \equiv z_n(\alpha, \varepsilon) \left[\lambda_{n\ell} + \alpha \Delta_n(\alpha, \lambda, \varepsilon)\right], \tag{4.3}$$

where $z_n(\alpha, \varepsilon) = 1+\mathcal{O}(\alpha)$ is the renormalization constant of $\mathcal{O}_{n\ell}(\varphi)$, viewed as a composite operator of the undeformed theory $\mathcal{R}$, while $\Delta_n(\alpha, \lambda, \varepsilon)$ depends only on $\lambda_{k\ell}$ with $k < n$ and is analytic in α. Indeed, we have proved in the previous section that every counterterm that renormalizes the product $\alpha^{p_{n\ell}}\lambda_{n\ell}$ carries a power $\alpha^{p_{n\ell}+L}$, where L is the number of loops. Thus the counterterms that renormalize $\lambda_{n\ell}$ are proportional to α^L, $L \geq 1$.

Inserting (3.12) into (4.3) and (4.2), and defining $\Delta_n(\alpha, \lambda, \varepsilon) = \lambda_\ell^n \overline{\Delta}_n(\alpha, f, \varepsilon)$, where $\overline{\Delta}_n(\alpha, f, \varepsilon)$ depends only on f_k with $k < n$, we get

$$f_n(\alpha, \varepsilon) = -\alpha\overline{\Delta}_n(\alpha, f, \varepsilon) + \zeta_n z_n^{-1}(\alpha, \varepsilon) Z_\ell^n(\alpha, \varepsilon). \tag{4.4}$$

This formula iteratively gives the functions f_n, $n > 1$, in terms of $\alpha, \zeta, \varepsilon$, and f_k with $k < n$. By direct differentiation, it is immediate to verify that the functions $f_n(\alpha, \varepsilon)$ of (4.4) do satisfy also (3.15). Thus, the solutions $f_n(\alpha, \xi, \varepsilon)$ of (3.15) must coincide with (4.4), once the constants ξ are written in terms of ζ, or vice versa. Observe that formula (4.4) shows again that the solution is analytic in α at $\varepsilon \neq 0$.

A quicker way to show that the bare relations (4.1) integrate the renormalized reduction equations (3.15) is as follows. Use (4.1) to write

$$\zeta_n = \frac{\lambda_{n\ell\mathrm{B}}}{\lambda_{\ell\mathrm{B}}^n}.$$

The right-hand side of this formula, rewritten in terms of renormalized couplings, is clearly independent of the renormalization point μ. Therefore ζ_n is an integral of motion of the RG flow. Since the renormalized reduction equations (3.15) are just ratios of RG equations (see (2.6)), $\lambda_{n\ell}(\lambda_\ell, \alpha)$ determined from (4.2) solves (3.15).

Relation between the bare and renormalized constants ζ, ξ and uniqueness of the doubly analytic reduction.

The particular solution (3.20) is analytic in α and ε. We now prove that, without paying attention to the α-dependence, the solution (3.20) is identified uniquely also by the requirement that both the bare and renormalized reduction relations be analytic in ε. For this purpose it is useful to work out the structure of the function $\zeta_n(\xi, \varepsilon)$.

First observe that

$$\lim_{\alpha\rightarrow 0} \overline{\Delta}_n(\alpha, f(\alpha, \xi, \varepsilon), \varepsilon) < \infty \qquad \text{at} \qquad \varepsilon \neq 0. \tag{4.5}$$

Indeed, every $f(\alpha, \xi, \varepsilon)$ is analytic in α at $\varepsilon \neq 0$ and $\overline{\Delta}_n(\alpha, f, \varepsilon)$ is analytic in α.

Applying (4.5) to (4.4) and recalling that z_n and Z_ℓ tend to one when α tends to zero, we obtain a useful formula to compute the relation between the bare constants ζ_n and the renormalized constants ξ_n, namely

$$\zeta_n(\xi, \varepsilon) = \lim_{\alpha\rightarrow 0} f_n(\alpha, \xi, \varepsilon), \qquad \varepsilon \neq 0. \tag{4.6}$$

Now, ζ_n is linear in ξ_n, so it is convenient to split it into the sum of two contributions,

$$\zeta_n(\xi, \varepsilon) = \overline{\zeta}_n(\xi, \varepsilon) + \xi_n\widehat{\zeta}_n(\varepsilon), \tag{4.7}$$

to be studied separately, such that $\overline{\zeta}_n(\varepsilon, \xi)$ depends on the ξ_k's with $k < n$.

Pole cancellations in (4.2).

The algebraic equations (4.2) contain poles. However, (4.4) solves (3.15), and we know that (3.15) admits a solution that is analytic in ε at $\alpha \neq 0$. Therefore, the poles of (4.2) have to mutually cancel out, once ζ_n is replaced by the appropriate function $\zeta_n(\xi, \varepsilon)$ (4.7). The mechanism of pole cancellation provides an alternative method to derive the results obtained in the previous section, including the invertibility conditions (3.19). We describe the cancellation in the case that is most interesting for our purposes, that is to say the doubly analytic solution $\overline{f}_n(\alpha, \varepsilon)$, which corresponds to $\xi = 0$, i.e. $\zeta_n = \zeta_n(0, \varepsilon) = \overline{\zeta}_n(0, \varepsilon) \equiv \overline{\zeta}_n(\varepsilon)$. We prove also that the function $\overline{\zeta}_n(\varepsilon)$ is analytic in ε.

First observe that the divergences of $\overline{\Delta}_n$, z_n and Z_ℓ are consistent with the renormalization-group. This is true also of $\overline{\Delta}_n(\alpha, \overline{f}(\alpha, \varepsilon), \varepsilon)$, when the functions f_k, $k < n$, are inductively replaced by the solutions $\overline{f}_k(\alpha, \varepsilon)$ of the reduction relations, because the reduction equations are just ratios of RG equations. Consequently, the double and higher poles of (4.4) are unambiguously related to the simple poles and it is sufficient to check that the simple poles of (4.4) cancel out to prove complete cancellation.

We proceed inductively. Assume that ζ_k, for $k < n$, are equal to analytic functions $\overline{\zeta}_k(\varepsilon)$ of ε such that $\overline{f}_k(\alpha, \varepsilon) \equiv -\alpha\overline{\Delta}_k(\alpha, \overline{f}, \varepsilon) + \overline{\zeta}_k(\varepsilon) z_k^{-1}(\alpha, \varepsilon) Z_\ell^k(\alpha, \varepsilon)$ are analytic in α and ε. Write $\overline{\zeta}_n(\varepsilon) = \sum_{k=0}^{\infty} \overline{\zeta}_{n,k}\varepsilon^k$. In (4.4) the coefficient $\overline{\zeta}_{n,k}$ is multiplied by a sum of objects of the form

$$\varepsilon^k \left(\frac{\alpha}{\varepsilon}\right)^m \alpha^r, \tag{4.8}$$

with $m, r \geq 0$. The simple pole is

$$\frac{1}{\varepsilon}\alpha^{1+k+r}. \tag{4.9}$$

Since $\overline{f}_k(\alpha, \varepsilon)$, $k < n$, are analytic in α and ε by the inductive hypothesis, the simple pole of $\overline{\Delta}_n$ is an analytic function of α. In total, the simple poles of (4.4) have the form

$$\frac{\alpha}{\varepsilon}\left(\sum_{s\geq 0} a_s \alpha^s + \sum_{k,r\geq 0} \overline{\zeta}_{n,k} c_{k,r} \alpha^{k+r}\right), \tag{4.10}$$

where a_s and $c_{k,r}$ are known numerical factors. If the coefficients of $\overline{\zeta}_{n,j}\alpha^j$ inside the parenthesis are nonzero it is possible to uniquely determine $\overline{\zeta}_{n,j}$ iteratively in j from the cancellation of the pole. The coefficient of $\overline{\zeta}_{n,j}\alpha^j$ depends only on the leading-log contributions to the wave-function renormalization constants, given by the standard formulas [17]

$$Z_\alpha = \left(1 - \frac{\beta_\alpha^{(1)}\alpha}{\varepsilon}\right)^{-1}, \qquad Z_\ell = Z_\alpha^{\gamma_\ell^{(1)}/\beta_\alpha^{(1)}}, \qquad z_n = Z_\alpha^{\gamma_{n\ell}^{(1)}/\beta_\alpha^{(1)}}. \tag{4.11}$$

Inside the parenthesis of (4.10) $\overline{\zeta}_{n,j}\alpha^j$ is multiplied by the coefficient

$$\frac{\left(-\beta_\alpha^{(1)}\right)^{j+1}}{(j+1)!}\prod_{i=0}^{j}\left(\frac{\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}} - i\right),$$

which must not vanish. Thus we recover the invertibility conditions (3.19) from pole cancellation. We conclude that the doubly analytic solution is determined by an ε-analytic constant $\zeta_n = \overline{\zeta}_n(\varepsilon)$.

Now we study the ξ-dependent terms of (4.7) and show that if the invertibility conditions (3.19) hold, when some ξ is nonzero either the renormalized or the bare relations are not analytic in ε. It is sufficient to prove the statement for $\xi_n \neq 0$ and $\xi_k = 0$ for $k < n$.

Comparing (3.18) with (4.4), and using (4.7), we obtain

$$\xi_n \overline{s}_n(\alpha, \varepsilon) = \xi_n \widehat{\zeta}_n(\varepsilon) z_n^{-1}(\alpha, \varepsilon) Z_\ell^n(\alpha, \varepsilon) = \xi_n \widehat{\zeta}_n(\varepsilon) s_n(\alpha, 0, \varepsilon),$$

where $s_n(\alpha, \alpha', \varepsilon)$ is given in (3.18). Now, in general $s_n(\alpha, 0, \varepsilon)$ is not analytic in ε, but $s_n(\alpha, \overline{\alpha}, \varepsilon)$ certainly is, if $\overline{\alpha} \neq 0$ and $\beta_\alpha(\alpha) \neq 0$. Formula (3.16) can be written as

$$f_n(\alpha, \xi, \varepsilon) = \overline{f}_n(\alpha, \varepsilon) + \xi_n \widehat{\zeta}_n(\varepsilon) s_n(\overline{\alpha}, 0, \varepsilon) s_n(\alpha, \overline{\alpha}, \varepsilon). \tag{4.12}$$

Assume that the invertibility conditions (3.19) are fulfilled. If $\widehat{\zeta}_n(\varepsilon)$ is an analytic function of ε , then the bare relations (4.1) are analytic, but the renormalized ones (4.12) are not, because $s_n(\overline{\alpha}, 0, \varepsilon)$ is not analytic in ε. The renormalized relations (4.12) become analytic choosing $\widehat{\zeta}_n(\varepsilon) = \widetilde{\zeta}_n(\varepsilon) s_n(0, \overline{\alpha}, \varepsilon)$, with $\widetilde{\zeta}_n(\varepsilon)$ analytic. Then, however, the bare relations (4.1) are not analytic.

Concluding, when the invertibility conditions are fulfilled, the bare and renormalized reduction relations are both analytic in ε if and only if $\xi_n = 0$, $\zeta_n = \overline{\zeta}_n(\varepsilon)$. This condition uniquely determines the reduction.

Renormalization mixing.

Taking into account of the renormalization mixing in the bare reduction, formulas (4.1) and (4.3) generalize to

$$\lambda^I_{n\ell\mathrm{B}} = Z^{IJ}_{n\ell}(\alpha, \lambda, \varepsilon) \lambda^J_{n\ell} \equiv \sum_J z_n^{IJ}(\alpha, \varepsilon) \left[\lambda^J_{n\ell} + \alpha \Delta^J_n(\alpha, \lambda, \varepsilon)\right] = \zeta^I_n \lambda^n_{\ell\mathrm{B}}, \tag{4.13}$$

where the coupling λ_ℓ is determined by the same equation (4.13) for $n = 1$, which we write as

$$\lambda^I_{\ell\mathrm{B}} = \sum_J z^{IJ} \lambda^J_\ell = \zeta^I \lambda_{\ell\mathrm{B}}. \tag{4.14}$$

Assume that the coefficient-matrix $\gamma_\ell^{(1)IJ}$ of the one-loop anomalous dimensions is arranged into its Jordan canonical form. With the same notational conventions as in the previous section, write $I = (\overline{I}, N)$, $\overline{I} = 1, \dots N-1$, $\gamma_\ell^{(1)\overline{I}N} = \gamma_\ell^{(1)N\overline{I}} = 0$ and assume that the eigenvalue $\gamma_\ell^{(1)NN} \equiv \gamma_\ell^{(1)}$ is real and has multiplicity one. Choose $\zeta^N = 1$ and define $\lambda^N_{\ell\mathrm{B}} = \lambda_{\ell\mathrm{B}} = Z_\ell \lambda_\ell$, $\lambda^N_\ell = \lambda_\ell$, $\lambda^{\overline{I}}_\ell = f^{\overline{I}} \lambda_\ell$. Then (4.14) gives

$$Z_\ell = z^{NN} + z^{N\overline{I}} f^{\overline{I}}, \qquad f^{\overline{I}} = (z^{-1})^{\overline{IJ}} \left(z^{NN} \zeta^{\overline{J}} + \zeta^{\overline{J}} z^{N\overline{K}} f^{\overline{K}} - z^{\overline{J}N}\right), \tag{4.15}$$

where $z^{NN} = 1 + \mathcal{O}(\alpha)$, $z^{N\overline{I}} = \mathcal{O}(\alpha^2)$, $z^{\overline{I}N} = \mathcal{O}(\alpha^2)$, $z^{\overline{IJ}} = \delta^{\overline{IJ}} + \mathcal{O}(\alpha)$.

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Writing $\lambda_{n\ell}^I(\alpha, \lambda_\ell, \varepsilon) = \lambda_\ell^n f_n^I(\alpha, \varepsilon)$, for $n > 1$, (4.4) is replaced by

$$f_n^I(\alpha, \xi, \varepsilon) = -\alpha \overline{\Delta}_n^I(\alpha, f, \varepsilon) + \sum_J \left(z_n^{-1}(\alpha, \varepsilon)\right)^{IJ} \zeta_n^J Z_\ell^n(\alpha, \varepsilon). \tag{4.16}$$

The doubly analytic solution can be worked out repeating the analysis of pole cancellation. The invertibility conditions are again that the matrices (3.27) have no integer eigenvalue for $n > 1$ and no positive integer eigenvalue for $n = 1$. The second equation of (4.15) is solved in powers of α, which determines the functions $f^{\overline{I}}$. Then the first formula of (4.15) gives Z_ℓ, which is inserted in (4.16). Finally, the equations (4.16) are solved for f_n^I, $n > 1$.

Repeating the arguments leading to (4.6), we find also

$$\zeta_n^I(\xi, \varepsilon) = \lim_{\alpha \to 0} f_n^I(\alpha, \xi_n, \varepsilon).$$

Again, when the invertibility conditions are fulfilled the bare and renormalized reduction relations cannot be both analytic in ε at $\xi_n \neq 0$.

Reduced subtraction scheme.

Observe that the reduction of couplings reduces also the scheme freedom. Indeed, a scheme change is a reparametrization in the space of couplings. Before the reduction, every coupling is independent and can be reparametrized independently. After the reduction, instead, only α and λ_ℓ are independent. Under an α, λ_ℓ-reparametrization the queue of the irrelevant deformation is reparametrized consistently. That is why, even if the minimal subtraction scheme is used for the unreduced theory, various evanescent terms appear after the reduction, due to the ε-dependence of the functions $\overline{f}_n(\alpha, \varepsilon)$. Explicit examples are given in the next sections.

Concluding, when the invertibility conditions are fulfilled, the reduced theory is described by the classical lagrangian (3.29), whose structure is preserved by renormalization. The divergences are subtracted away with field redefinitions and a finite number of independent renormalization constants: those belonging to the renormalizable sector $\mathcal{L}_{\mathcal{R}}[\varphi, g]$ plus a renormalization constant for the coupling λ_ℓ that multiplies the head of the deformation. The bare lagrangian reads

$$\mathcal{L}_{\mathrm{B}}[\varphi_{\mathrm{B}}] = \mathcal{L}_{\mathcal{R}}[\varphi_{\mathrm{B}}, g_{\mathrm{B}}] + \sum_{n=1}^{\infty} \sum_I g_{\mathrm{B}}^{2p_{n\ell}^I} \overline{\zeta}_n^I(\varepsilon) \lambda_{\ell\mathrm{B}}^n \, \mathcal{O}_{n\ell}^I(\varphi_{\mathrm{B}}). \tag{4.17}$$

Formulas (3.29) and (4.17) define *the* fundamental interaction having head $\sum_I g^{2p_\ell^I} \lambda_\ell^I \mathcal{O}_\ell^I(\varphi)$.

We have also seen that the invertibility conditions for the existence of the reduction to all orders are captured just by the leading-log approximation. Remarkably, the leading-log approximation can be solved exactly and provides a good illustration of the properties found so far. This calculation is done in the section 6.

5. Physical invertibility conditions in the absence of three-leg marginal vertices

When there are no three-leg vertices, e.g. $\mathcal{R}$ is the theory φ^4 in four dimensions (but similar arguments apply if $\mathcal{R}$ is the theory φ^6 in three dimensions), it is possible to refine the previous analysis and recover the results of [4, 3]. It is useful to introduce the numbers $\widetilde{p}_n \equiv p_{n\ell} - np_\ell$, which are always integers, by U-parity. Indeed, if the head $\mathcal{O}_\ell(\varphi)$ is U-even, then the terms of the queue are U-even, so the $\widetilde{p}_n$'s are integers. If the head is U-odd, then the terms of the queue of levels $n\ell$ with n odd are U-odd, while those with n even are U-even. In both cases, $\widetilde{p}_n$ is integer.

Define the integer

$$q_k \equiv \max(-k + 1 - \widetilde{p}_k, 0). \tag{5.1}$$

We prove that the general form of $\overline{\zeta}_n(\varepsilon)$ is

$$\overline{\zeta}_n(\varepsilon) = \varepsilon^{q_n} \sum_{k=0}^{\infty} \overline{\zeta}_{n,k} \varepsilon^k, \tag{5.2}$$

while the general form of $\overline{f}_k(\alpha, \varepsilon)$ is

$$\overline{f}_k(\alpha, \varepsilon) = \alpha^{q_k} \sum_{j=0}^{q_k} \overline{f}_{j,k}(\alpha, \varepsilon) \left(\frac{\varepsilon}{\alpha}\right)^j, \tag{5.3}$$

where $\overline{f}_{j,k}(\alpha, \varepsilon)$ are analytic functions of α and ε. Formula (5.3) shows that the α -powers smaller than $-k + 1 - \widetilde{p}_k$, if any, belong only to the evanescent sector.

The proof is done by induction, studying the pole cancellation in (4.4). Assume that (5.3) holds for $k < n$. A theorem derived in ref. [4] states that the maximal ε-pole of a Feynman diagram with V vertices and L loops is at most of order equal to $\min(V - 1, L)$. Then the contributions to $\overline{\Delta}_n(\alpha, f(\alpha, \varepsilon), \varepsilon)$ from the diagrams G with n_k irrelevant vertices of level k, L loops, v_4 marginal four-leg vertices and $V = v_4 + \sum_{k<n} n_k$ vertices have the form

$$\begin{aligned}\overline{\Delta}_n(\alpha, f(\alpha, \varepsilon), \varepsilon) &= \sum_G \frac{\varepsilon^s}{\varepsilon^{\min(V-1,L)}} \alpha^{L-1} \prod_{k<n} \overline{f}_k^{n_k}(\alpha, \varepsilon) \\ &= \sum_G \frac{\varepsilon^{s'}}{\varepsilon^{\min(V-1,L)}} \alpha^{L-1+t+\sum_{k<n} n_k q_k} \prod_{k<n} \left(\frac{\varepsilon}{\alpha}\right)^{j_k},\end{aligned} \tag{5.4}$$

where s, s', t, j_k are non-negative integers and $j_k \le n_k q_k$ and $\sum_{k<n} k n_k = n$. The factor α^{L-1} comes from the α -powers attached to the vertices, as shown in the previous sections. The factors ε^s, $\varepsilon^{s'}$ are inserted in (5.4) to take care of the subleading divergences and the extra powers of ε coming from $\overline{f}_{j,k}(\alpha, \varepsilon)$, while α^t takes care of the extra powers of α coming from $\overline{f}_{j,k}(\alpha, \varepsilon)$. The coefficients of the sum are not important for our analysis and so are left unspecified. Specializing to the simple poles we get contributions of the type

$$\frac{1}{\varepsilon} \alpha^{L-\min(V-1,L)+\sum_{k<n} n_k q_k + t + s'}$$

Formula (3.6) can be written as

$$v_4 + \sum_{k<n} n_k \widetilde{p}_k = \widetilde{p}_n + L.$$

Using this relation and (5.1), it is immediate to derive the inequality

$$L - \min(V-1, L) + \sum_{k<n} n_k q_k \geq q_n, \tag{5.5}$$

so the α-exponent of the simple pole of $\overline{\Delta}_n$ is always $\geq q_n$, and the simple poles contained in $\overline{\Delta}_n(\alpha, \overline{f}(\alpha, \xi, \varepsilon), \varepsilon)$ are multiplied by powers α^{q_n+s}, $s \geq 0$.

With the ansatz (5.2), in (4.4) the coefficient $\overline{\zeta}_{n,k}$ is multiplied by a sum of objects of the form

$$\varepsilon^{q_n+k}\left(\frac{\alpha}{\varepsilon}\right)^m \alpha^r, \tag{5.6}$$

with $m, r \geq 0$. The simple pole is

$$\frac{1}{\varepsilon}\alpha^{q_n+1+k+r}.$$

In total, the simple poles of (4.4) have the form

$$\frac{\alpha^{q_n+1}}{\varepsilon}\left(\sum_{s\geq 0} a_s \alpha^s + \sum_{k,r\geq 0} \overline{\zeta}_{n,k} c_{k,r} \alpha^{k+r}\right), \tag{5.7}$$

where a_s and $c_{k,r}$ are known numerical factors. Thus, if the coefficients of $\overline{\zeta}_{n,j}\alpha^j$ are nonzero it is possible to determine $\overline{\zeta}_{n,j}$ iteratively in j from the cancellation of the pole. Finally, using (4.11) the term $\overline{\zeta}_{n,j}\alpha^j$ inside the parenthesis of (5.7) is multiplied by the coefficient

$$\frac{\left(-\beta_\alpha^{(1)}\right)^{q_n+j+1}}{(q_n+j+1)!}\prod_{i=0}^{q_n+j}\left(\frac{\gamma_{n\ell}^{(1)} - n\gamma_\ell^{(1)}}{\beta_\alpha^{(1)}} - i\right),$$

thus the invertibility conditions are again (3.19).

Once the poles have been cancelled out and the constants $\overline{\zeta}_{n,k}$ are determined, collecting (5.4) and (5.6) we obtain

$$\overline{f}_n \sim \alpha^{q_n}\left[\sum_{\substack{L\geq 1,\, u,s,i_n\geq 0\\ L+i_n\leq q_n+s'+u}} \left(\frac{\varepsilon}{\alpha}\right)^{q_n-L-i_n}\alpha^t\varepsilon^{s'+u} + \sum_{\substack{m,r,j\geq 0\\ m\leq q_n+j}} \overline{\zeta}_{n,j}\varepsilon^j\alpha^r\left(\frac{\varepsilon}{\alpha}\right)^{q_n-m}\right], \tag{5.8}$$

having written $\sum_{k<n} j_k = \sum_{k<n} q_k n_k - i_n$, $i_n \geq 0$, and $\sum_{k<n} q_k n_k - \min(V-1, L) = q_n - L + u$, $u \geq 0$ (see (5.5)). We see that $\overline{f}_n(\alpha, \varepsilon)$ has the form (5.3) for $k = n$, which reproduces the inductive hypothesis.

In the absence of three-leg vertices, the invertibility conditions found in [4, 3] read in the notation of this paper (recall that here $\gamma_{n\ell}$ denotes the anomalous dimensions of the operators $g^{2p_n}\mathcal{O}_n(\varphi)$)

$$r_n \equiv \tau_n + n - 1 + \widetilde{p}_n \notin \mathbb{N}. \tag{5.9}$$

These invertibility conditions can be more or less restrictive than the regularized invertibility conditions (3.19), but they are not in contradiction with (3.19) in the physical limit.

Recall that both (5.9) and (3.19) are sufficient, but not necessary conditions and in particular cases they can be relaxed. If $n-1+\widetilde{p}_n>0$ the conditions (3.19) are less restrictive than (5.9). In this case the violations of (5.9) that are not violations of (3.19) do not cause the introduction of any new parameters. If $n-1+\widetilde{p}_n<0$ the conditions (3.19) are more restrictive than (5.9). Then there are situations where (3.19) are violated but (5.9) are fulfilled. This happens if $\tau_n=N\in\mathbb{N}$ and $r_n=N-q_n<0$. In this case it is necessary to introduce new independent parameters, which however affect only the evanescent sector, with no physical consequences. Indeed, formula (5.3) for $k=n$ shows that the physical function $\overline{f}_n(\alpha,0)$ is not interested by the violation of the invertibility condition, because it starts with the power α^{q_n}, while the invertibility problem occurs only when the power α^N is present. Thus the $\varepsilon=0$ results of [3, 4] and the $\varepsilon\neq 0$ results of [4] are fully recovered.

6. Explicit leading-log solution

In this section we solve the infinite reduction at the leading-log level. We show that the leading-log approximation is sufficient to derive the invertibility conditions for the existence of the infinite reduction to all orders.

At the leading-log level, the beta function of the marginal coupling α

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu}=\widehat{\beta}_\alpha(\alpha,\varepsilon)=\beta_\alpha(\alpha)-\varepsilon\alpha=-\varepsilon\alpha+\beta_1\alpha^2+\mathcal{O}(\alpha^3) \tag{6.1}$$

and the anomalous dimensions

$$\gamma_{n\ell}(\alpha)=\gamma_{n\ell}^{(1)}\alpha+\mathcal{O}(\alpha^2)\,,$$

can be truncated to one loop. The models with and without three-leg marginal vertices can be treated with a unique formalism, using formula (5.3), where it is understood that, in the presence of three-leg marginal vertices q_k is just equal to zero. The leading-log approximation amounts to take the lowest α-powers at $\varepsilon=0$ and the corresponding α powers at higher orders in ε. More precisely, in this approximation $\overline{f}_k(\alpha,\varepsilon)$ and $\overline{\delta}_n(\alpha,\varepsilon)$ read

$$\overline{f}_k(\alpha,\varepsilon)=\alpha^{q_k}\sum_{j=0}^{q_k}\overline{f}_{j,k}\left(\frac{\varepsilon}{\alpha}\right)^j,\qquad \overline{\delta}_n(\alpha,\varepsilon)=\alpha^{q_n}\sum_{j=0}^{q_n}d_{j,n}\left(\frac{\varepsilon}{\alpha}\right)^j$$

where q_n is an integer and $\overline{f}_{j,k}$, $d_{j,n}$ are constants. The solution (3.17) is

$$\begin{aligned} f_n(\alpha,\xi_n,\varepsilon) = & -\frac{\alpha^{q_n}}{\beta_\alpha^{(1)}}\sum_{j=0}^{q_n}\frac{d_{j,n}\varepsilon^j}{\alpha^j(\tau_n-q_n+j)}\;{}_2F_1\left[1,j-q_n,\tau_n-q_n+j+1,\frac{\varepsilon}{\alpha\beta_\alpha^{(1)}}\right]+\\ & +\xi_n\left(\alpha\beta_\alpha^{(1)}-\varepsilon\right)^{\tau_n}. \end{aligned} \tag{6.2}$$

Observe that the hypergeometric functions appearing in the sum are polynomial, since q_n-j is a non-negative integer.

At the level of bare couplings, the reduction has the form (4.1). Manipulating the formulas given above and using (4.6), the formula for $\zeta_n(\xi_n, \varepsilon)$ can be derived. The result is

$$\zeta_n(\xi_n,\varepsilon) = \varepsilon^{q_n} \sum_{j=0}^{q_n} d_{j,n} \left(-\beta_\alpha^{(1)}\right)^{j-q_n-1} \frac{\Gamma(\tau_n - q_n + j)\,\Gamma(q_n - j + 1)}{\Gamma(\tau_n + 1)} + \xi_n(-\varepsilon)^{\tau_n}. \tag{6.3}$$

We see again that if $\tau_n \notin \mathbb{N}$ the relation $\zeta_n(\xi_n, \varepsilon)$ is not analytic in ξ_n and ε . Both the bare and renormalized reduction relations are analytic in ε only for $\xi_n = 0$, which gives

$$\overline{\zeta}_n(\varepsilon) = \varepsilon^{q_n} \sum_{j=0}^{q_n} d_{j,n} \left(-\beta_\alpha^{(1)}\right)^{j-q_n-1} \frac{\Gamma(\tau_n - q_n + j)\,\Gamma(q_n - j + 1)}{\Gamma(\tau_n + 1)}. \tag{6.4}$$

This formula uniquely determines the reduction.

For $q_n = 0$ we simply have

$$f_n(\alpha, \xi_n, \varepsilon) = -\frac{d_{0,n}}{\tau_n \beta_\alpha^{(1)}} + \xi_n \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)^{\tau_n}, \qquad \overline{\zeta}_n(\varepsilon) = -\frac{d_{0,n}}{\tau_n \beta_\alpha^{(1)}}.$$

Violations of the invertibility conditions.

It is interesting to describe the appearance of new parameters, when the invertibility conditions are violated, in the leading-log approximation. Assume that some regularized invertibility conditions (3.19) are violated, i.e. $\tau_n = \overline{r} \in \mathbb{N}$. To study this situation it is convenient to approach it continuously from $\tau_n = \overline{r} + \delta$ and then take the limit $\delta \to 0$. If $\overline{r} > q_n$ this limit is trivial in the leading-log approximation, so we just need to discuss the case $\overline{r} \leq q_n$.

Collecting the singular terms of (6.2) we get an expression of the form

$$f_n(\alpha, \xi_n, \varepsilon) = \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)^{\overline{r}} \left\{ \frac{a}{\delta} \varepsilon^{q_n - \overline{r}} + \xi_n \left[1 + \delta \ln \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)\right]\right\} + \alpha^{q_n} P_n(\varepsilon/\alpha) + \mathcal{O}(\delta, \xi_n \delta^2),$$

where a is a known numerical factor and $P_n(\varepsilon/\alpha)$ is a certain ξ- and δ-independent polynomial of degree q_n. The δ-singularity can be removed redefining ξ_n as

$$\xi_n = -\frac{a}{\delta}\varepsilon^{q_n - \overline{r}} + \xi_n',$$

thus obtaining a non-singular expression

$$f_n(\alpha, \xi_n, \varepsilon) = \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)^{\overline{r}} \left\{ \xi_n' - a\varepsilon^{q_n - \overline{r}} \ln \left(\alpha\beta_\alpha^{(1)} - \varepsilon\right)\right\} + \alpha^{q_n} P_n(\varepsilon/\alpha).$$

Finally, the relation between the bare and renormalized constants ζ_n and ξ_n' at $\zeta_k = 0$, $k < n$, read

$$\zeta_n(\xi, \varepsilon) = \lim_{\alpha \to 0} f_n(\alpha, \xi_n, \varepsilon) = (-\varepsilon)^{\overline{r}} \left[\xi_n' - a\varepsilon^{q_n - \overline{r}} \ln(-\varepsilon) + b\varepsilon^{q_n - \overline{r}}\right], \tag{6.5}$$

where b is another known numerical factor, originated by $\alpha^{q_n} P_n(\varepsilon/\alpha)$. We see that no choice of the constant ξ_n' is able to remove the analyticity violation in both the bare and renormalized reduction relations. The violation can be hidden in a new independent coupling, but since $\ln(-\varepsilon)$ is multiplied by $\varepsilon^{q_n - \overline{r}}$ it is sufficient to write $\xi_n' = \varepsilon^{q_n - \overline{r}} \xi_n''$ and associate the new coupling with ξ_n''. Therefore the new coupling belongs to the evanescent sector if $\overline{r} < q_n$, it is physical if $\overline{r} = q_n$.

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7. Irrelevant deformations in the presence of several marginal couplings

In this section we describe the construction of irrelevant deformations when the renormalizable subsector $\mathcal{R}$ contains more independent marginal couplings.

Consider a renormalizable theory with two marginal couplings α and $\rho = \alpha\eta$. It is convenient, for intermediate purposes, to express η as a function $\widetilde{\eta}(\alpha, \xi, \varepsilon)$ of α and an arbitrary constant ξ, as explained in section 2, solving the equations (2.6). Then the marginal sector is described by a unique running marginal coupling, α, plus an arbitrary constant, and most of the arguments of the infinite reduction proceed as in the presence of a single marginal coupling. As before, we state two equivalent criteria for the infinite reduction. The first criterion is based on the analyticity properties of the renormalized reduction relations. The second criterion is based on the comparison between the analyticity properties of the renormalized and bare reduction relations.

Precisely, when certain invertibility conditions, derived below, hold, the reduction is uniquely determined by the requirement that

1) the renormalized reduction relations be perturbatively meromorphic in α, analytic in η and analytic in ε;

or, equivalently, by the requirement that

2) the renormalized reduction relations be analytic in ε and ξ and at the same time the bare relations be analytic in ε and $\xi(-\varepsilon)^Q$, where Q is a non-integer exponent that can possibly depend on ε.

The analyticity requirements in ξ and $\xi(-\varepsilon)^Q$ are obviously due to the presence of the second marginal coupling.

We focus on the leading-log approximation, for simplicity, whose beta functions (2.4) have the one-loop coefficients (2.12). For definiteness, we choose the positive sign in front of s in the leading-log solution (2.13). We study the 2ℓ-level terms belonging to the queue of the irrelevant deformation, using the minimal parametrization

$$\lambda_\ell \mathcal{O}_\ell(\varphi) + \lambda_{2\ell}\mathcal{O}_{2\ell}(\varphi) + \cdots,$$

and assuming that the beta functions of the irrelevant couplings are $\widehat{\beta}_\ell = \beta_\ell - \varepsilon\lambda_\ell$, $\widehat{\beta}_{2\ell} = \beta_{2\ell} - \varepsilon\lambda_{2\ell}$, in the minimal subtraction scheme. For α small the lowest-order beta functions of λ_ℓ and $\lambda_{2\ell}$ have generically the forms

$$\beta_\ell = \lambda_\ell \alpha(d + e\eta), \qquad \beta_{2\ell} = \lambda_{2\ell}\alpha(f + g\eta) + h\lambda_\ell^2, \tag{7.1}$$

where d, e, f, g, h are numerical factors. The coupling $\lambda_{2\ell}$ is related to λ_ℓ and α, η by a relation of the form

$$\lambda_{2\ell} = f_2(\alpha, \eta, \varepsilon)\lambda_\ell^2. \tag{7.2}$$

The function $f_2(\alpha, \eta, \varepsilon)$ can be worked out using a procedure similar to the one described in appendix A for multivariable renormalization constants. Define the function

$$\widetilde{f}_2(\alpha, \xi, \varepsilon) = f_2\left(\alpha, \widetilde{\eta}(\alpha, \xi, \varepsilon), \varepsilon\right). \tag{7.3}$$

Differentiating (7.2) and using (7.1), we find the equation

$$\widetilde{\beta}_\alpha\left(\alpha, \widetilde{\eta}(\alpha,\xi,\varepsilon),\varepsilon\right)\ \frac{\mathrm{d}\widetilde{f}_2(\alpha,\xi,\varepsilon)}{\mathrm{d}\alpha} + 2\alpha \widetilde{f}_2(\alpha,\xi,\varepsilon)\left(\widetilde{d} + \widetilde{e}\widetilde{\eta}(\alpha,\xi,\varepsilon)\right) - \varepsilon \widetilde{f}_2(\alpha,\xi,\varepsilon) = h, \quad (7.4)$$

where $\widetilde{d} = d - f/2$ and $\widetilde{e} = e - g/2$. The solution depends on ξ and a further arbitrary constant k_2. Eliminating ξ with the help of (2.15), the solution reads

$$f_2(\alpha,\eta,\varepsilon,k_2) = \overline{f}_2(\alpha,\eta) + k_2\ s_2(\alpha,\eta,\varepsilon), \quad (7.5)$$

where

$$\overline{f}_2(\alpha,\eta) = \frac{h(1-z)}{\alpha s(\gamma-1)}\ {}_2F_1[1,\gamma-2\widetilde{e}/c,\gamma,z], \qquad s_2(\alpha,\eta,\varepsilon) = \frac{1}{\alpha} z^{1-\gamma}(1-z)^{2\widetilde{e}/c}, \quad (7.6)$$

with

$$\gamma = 1 + \frac{\widetilde{e}}{c} + \frac{1}{s}\left(2\widetilde{d} - \beta_1 - b\frac{\widetilde{e}}{c}\right),$$

and $z = \xi(\beta_1\alpha - \varepsilon)^{s/\beta_1}$. The quantity k_2 is constant along the RG flow, and can be viewed as a function of ξ and ε.

The analyticity properties of the solution can be analyzed in the α-ξ parametrization. The function $\widetilde{\eta}$ of (2.13) is analytic in α, ε and $z = \xi(\beta_1\alpha - \varepsilon)^{s/\beta_1}$. The special solution $\overline{f}_2(\alpha,\eta)$, on the other hand, is meromorphic in α, analytic in ε and analytic in z at $z = 0$, so it satisfies the requirement 1) stated above.

To study the arbitrariness of the general solution $f_2(\alpha,\eta,\varepsilon)$ write, for example, $k_2(\xi,\varepsilon) = k_2'\xi^p\varepsilon^q$. If

$$(1-\gamma)\frac{s}{\beta_1} = m + n\frac{s}{\beta_1}, \qquad m,n \in \mathbb{N}, \quad (7.7)$$

then taking $p = n - 1 + \gamma$ and $q \in \mathbb{N}$ we have

$$k_2(\xi)\ s_2(\alpha,\eta,\varepsilon) = k_2'\frac{1}{\alpha}\varepsilon^q(\beta_1\alpha - \varepsilon)^m\left(\xi(\beta_1\alpha-\varepsilon)^{s/\beta_1}\right)^n(1-z)^{2\widetilde{e}/c},$$

which is meromorphic in α with the right behavior at $\alpha \sim 0$, analytic in ε and in $\xi(\beta_1\alpha - \varepsilon)^{s/\beta_1}$. Thus the invertibility conditions for requirement 1) are that there should exist no pair of integers m, n such that (7.7) holds. Assuming that s is irrational or complex, which happens in most cases, and recalling that the ratios of one-loop coefficients are rational numbers, the invertibility conditions are equivalent to

$$-\frac{\widetilde{e}}{c} \notin \mathbb{N} \qquad \text{or} \qquad 1 + \frac{b\widetilde{e}}{c\beta_1} - \frac{2\widetilde{d}}{\beta_1} \notin \mathbb{N}. \quad (7.8)$$

It is sufficient to fulfill one of the two conditions (7.8) to fix $k_2' = 0$ and uniquely determine the reduction by requirement 1).

The special solution $\overline{f}_2(\alpha,\eta)$ is analytic in $z = 0$. Using the k_2-freedom it is possible to have special solutions that are analytic in $z = 1$ or $z = \infty$ [3]. The existence conditions around $z = \infty$ are

$$\left(1 - \gamma + 2\frac{\widetilde{e}}{c}\right)\frac{s}{\beta_1} \neq m - n\frac{s}{\beta_1}, \qquad m,n \in \mathbb{N},$$

which are equivalent to (7.8) if s is irrational or complex. Finally, around $z = 1$ the invertibility conditions are just

$$2\frac{\widetilde{e}}{c} - 1 \notin \mathbb{N}. \tag{7.9}$$

Now, let us study the reduction at the bare level. The bare reduction relations have the form

$$\lambda_{2\ell\mathrm{B}} = \zeta_2(\eta_{\mathrm{B}}, \varepsilon)\frac{\lambda_{\ell\mathrm{B}}^2}{\alpha_{\mathrm{B}}}. \tag{7.10}$$

The powers of $\lambda_{\ell\mathrm{B}}$ and α_{B} are fixed matching the dimensionalities at $\varepsilon \neq 0$ and demanding analyticity or meromorphy. The relation contains an arbitrary function of the dimensionless bare coupling η_{B}.

Rewrite (7.10) in terms of the renormalized couplings,

$$f_2 Z_{2\ell} = \frac{\zeta_2 Z_\ell^2}{\alpha Z_\alpha}, \tag{7.11}$$

then use (2.13) to express this equality in terms of α, ξ , and ε. Taking the limit $\alpha \to 0$, where all Z s tend to one, the formula

$$\zeta_2 = \lim_{\alpha\to 0} \alpha \widetilde{f}_2(\alpha, \xi, \varepsilon, k_2)$$

is obtained, which allows us to compute the constant ζ_2 as a function of ξ, k_2 and ε. The result is

$$\zeta_2 = \frac{h(1-z_{\mathrm{B}})}{s(\gamma-1)}\ {}_2F_1[1, \gamma - 2\widetilde{e}/c, \gamma, z_{\mathrm{B}}] + k_2\ z_{\mathrm{B}}^{1-\gamma}(1-z_{\mathrm{B}})^{2\widetilde{e}/c}, \tag{7.12}$$

where $z_{\mathrm{B}} = \xi(-\varepsilon)^{s/\beta_1}$.

Now we show that the reduction is uniquely fixed also demanding that the renormalized relations be analytic in ε and ξ and, at the same time, the bare relations be analytic in ε and $\xi(-\varepsilon)^{s/\beta_1}$, without paying attention to the α -dependence. We have shown in section 2 that at $\alpha \neq 0$ the function $\widetilde{\eta}(\alpha, \xi, \varepsilon)$ is analytic in ξ and ε. With arguments similar to the ones of section 2 it is possible to define the arbitrary constant k_2 of $\widetilde{f}_2(\alpha, \xi, \varepsilon, k_2) \equiv f_2(\alpha, \widetilde{\eta}(\alpha, \xi, \varepsilon), \varepsilon, k_2)$ in such a way that $\widetilde{f}_2$ is analytic in ε and ξ. In formula (7.5) this can be achieved writing $k_2 = k_2'\xi^{n-1+\gamma}\varepsilon^q$ with $n, q \in \mathbb{N}$. However, if $q > 0$ the arbitrariness affects only the evanescent sector, with no observable consequence, so we can take $q = 0$. Then (7.12) immediately shows that the bare relations are analytic in ε and $\xi(-\varepsilon)^{s/\beta_1}$ precisely when (7.7) holds, so the invertibility conditions are again (7.8). We conclude that the criterion 2) is equivalent to the criterion 1).

The same invertibility conditions are found demanding that the renormalized relations be analytic in ε and ξ, and the bare relations be analytic in ε and $\xi(-\varepsilon)^{-s/\beta_1}$. Finally, the reduction is uniquely fixed also demanding that the renormalized relations be analytic in ε and ξ, and the bare relations be analytic in ε and $1 - \xi(-\varepsilon)^{s/\beta_1}$, in which case the invertibility conditions are (7.9).

Beyond the leading-log approximation, the α-η parametrization can be used in connection with formula (2.20). Alternatively, it is convenient to use the variables u' and v

defined in section 2, and related quantities. Then for example analyticity in $\xi(-\varepsilon)^{s/\beta_1}$ is replaced by analyticity in $\xi(-\varepsilon)^{Q}$, where Q is defined in equation (2.31), and so on.

The generalization to theories with more marginal couplings follows the same guidelights and is left to the reader.

8. Conclusions

The field-theoretical investigation of non-renormalizable interactions can clarify some aspects of quantum field theory that have so far been underestimated and allows us to explore new ideas for quantum gravity and, more generally, physics beyond the Standard Model. At the conceptual level, there is no strict physical reason why a theory should be discarded just because it is not power-counting renormalizable. In this paper and related ones we have shown that there is not even a *practical* reason to justify its exclusion: in a wide class of non-renormalizable theories calculations are doable in a perturbative fashion and the number of independent couplings can be kept finite consistently with renormalization.

On the other hand, we cannot forget that power-counting renormalizability has been the main guidelight to build the Standard Model. Vector bosons and the Higgs field have been introduced to cure the non-renormalizability of the Fermi theory of weak interactions. The vector bosons have already been seen, while the Higgs field will hopefully be discovered at LHC. In view of these successes, it is hard to deny an important role to power-counting renormalizability in quantum field theory. Nevertheless, it is mandatory to understand such a role more precisely. Presumably, power-counting renormalizability is a preliminary, imperfect version of a deeper selection principle still well hidden in quantum field theory. The final version of this principle should leave room also for quantum gravity and new physics beyond the Standard Model. We hope that our investigations can help uncovering the final form of the selection principle.

In this paper we have studied the reduction of couplings for renormalizable and non-renormalizable theories at the regularized level. The dimensional-regularization technique is the most convenient framework to prove all-order theorems and have control on infinitely many lagrangian terms. It is possible to formulate the rules of the infinite reduction comparing the reduction relations at the renormalized and bare levels. If suitable invertibility conditions are fulfilled, the infinite reduction is uniquely determined by the contemporary ε-analyticity of the renormalized and bare relations. When the invertibility conditions are violated, new couplings appear along the way. It is possible to count the parameters of the non-renormalizable interaction, or study their distribution and density, just from knowledge of the renormalizable subsector $\mathcal{R}$, before turning the non-renormalizable interaction on.

We have mainly worked with theories where $\mathcal{R}$ contains a unique marginal coupling, but with some additional effort the results can be generalized to theories containing more marginal couplings. The marginal sector has to be fully interacting and it is not possible to switch the marginal interactions off keeping the irrelevant interaction on. Generalizations to theories with a free or partially interacting marginal sector demand further insight, in view of applications to quantum gravity in four dimensions.

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A. Derivation of the renormalization constants from the beta functions

In this appendix I describe a general procedure to derive multivariable renormalization constants from their beta functions, which is useful for several arguments of the paper.

In the minimal subtraction scheme the renormalization constants Z_α satisfy RG equations of the form

$$\frac{\mathrm{d}\ln Z_\alpha}{\mathrm{d}\ln\mu} = -\frac{\beta_\alpha}{\alpha}. \tag{A.1}$$

When the theory has only one coupling α, it is possible to write $Z_\alpha = Z_\alpha(\alpha, \varepsilon)$ and immediately integrate the RG equation

$$\frac{\mathrm{d}\ln Z_\alpha}{\mathrm{d}\alpha} = -\frac{\beta_\alpha(\alpha)}{\alpha\widehat{\beta}_\alpha(\alpha,\varepsilon)}, \qquad\qquad Z_\alpha = \exp\left(-\int_0^\alpha \frac{\beta_\alpha(\alpha')\mathrm{d}\alpha'}{\alpha'\widehat{\beta}_\alpha(\alpha',\varepsilon)}\right). \tag{A.2}$$

The integral in (A.2) is well-defined at $\varepsilon \neq 0$.

More generally, let $\alpha, \lambda_1, \cdots \lambda_n$ denote the couplings in the minimal parametrization. The problem we want to solve is to reconstruct the renormalization constants $Z_\alpha(\alpha, \lambda_i, \varepsilon)$ and $Z_i(\alpha, \lambda_i, \varepsilon)$ from the beta functions of α and λ_i.

The RG equations have the form

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu} = \widehat{\beta}_\alpha(\alpha,\lambda_i,\varepsilon) = \beta_\alpha(\alpha,\lambda_i) - p\varepsilon\alpha, \qquad \frac{\mathrm{d}\lambda_i}{\mathrm{d}\ln\mu} = \widehat{\beta}_i(\alpha,\lambda_i,\varepsilon) = \beta_i(\alpha,\lambda_i) - p_i\varepsilon\lambda_i, \tag{A.3}$$

where p, p_i are related to the numbers N, N_i of legs of the respective vertices by formula (3.3): $p = N/2-1$, $p_i = N_i/2-1$. The bare couplings are

$$\alpha_{\mathrm{B}} = \mu^{p\varepsilon}\alpha Z_\alpha(\alpha,\lambda_i,\varepsilon), \qquad \lambda_{i\mathrm{B}} = \mu^{p_i\varepsilon}\lambda_i Z_i(\alpha,\lambda_i,\varepsilon).$$

Since a non-trivial theory contains at least a vertex with three legs or more, we can assume that p is strictly positive. Instead of α, it is convenient to use the variable $\alpha' = \alpha^{1/p}$, which has $p' = 1$. We then suppress the primes on α and p, which is equivalent to assume that α is defined so that it has $p = 1$. We are not assuming that the marginal sector contains a single marginal coupling, nor that the theory has an interacting marginal sector. It is convenient, but not necessary, to assume that α is a marginal coupling.

Next, it is convenient to define non-minimal couplings η_i such that

$$\lambda_i = \alpha^{p_i}\eta_i, \qquad \frac{\mathrm{d}\eta_i}{\mathrm{d}\ln\mu} = \overline{\beta}_i(\alpha,\eta_i), \qquad \frac{\mathrm{d}\alpha}{\mathrm{d}\ln\mu} = \overline{\beta}_\alpha(\alpha,\eta_i) - \varepsilon\alpha. \tag{A.4}$$

Write their bare couplings as

$$\alpha_{\mathrm{B}} = \mu^\varepsilon\alpha\overline{Z}_\alpha(\alpha,\eta_i,\varepsilon), \qquad \eta_{i\mathrm{B}} = \eta_i + \alpha\Delta_i(\alpha,\eta_i,\varepsilon). \tag{A.5}$$

Diagrammatic arguments analogous to those of sections 2 and 3 (see formulas (2.2) and (3.6)), allow us to prove that $\overline{Z}_\alpha(\alpha,\eta_i,\varepsilon)-1$, $\overline{\beta}_i(\alpha,\eta_i)$ and $\alpha\Delta_i(\alpha,\eta_i,\varepsilon)$ are analytic in α and their L-loop contributions are of order α^L. A quicker argument proceeds as follows.

It is sufficient to consider the bare Feynman diagrams, since the renormalized diagrams inherit their α-structure from the bare ones. By construction, in the non-minimal

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parametrization (A.4) $\alpha_{\rm B}$ is the unique bare coupling that has a non-vanishing dimensionality-defect, equal to one. Now, each loop carries a momentum integral

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d},$$

where d is the continued spacetime dimension. That means that each loop carries a dimensionality-defect equal to -1, which has to be compensated by a factor $\alpha_{\rm B}$. Therefore a power $\alpha_{\rm B}^L$ is associated with each L-loop integral, as claimed.

Note that the L-loop contributions to $\overline{\beta}_\alpha(\alpha,\eta_i)$ are of order α^{L+1}.

Define the functions $\widetilde{\eta}_i(\alpha,\xi,\varepsilon)$ as the solutions of the differential equations

$$\frac{\mathrm{d}\widetilde{\eta}_i}{\mathrm{d}\alpha}=\frac{\overline{\beta}_i(\alpha,\widetilde{\eta}_i)}{\overline{\beta}_\alpha(\alpha,\widetilde{\eta}_i)-\varepsilon\alpha}, \tag{A.6}$$

ξ_i denoting the arbitrary constants, or, equivalently, as the solutions of the algebraic bare reduction relations

$$\eta_i+\alpha\Delta_i(\alpha,\eta_i,\varepsilon)=\zeta_i, \tag{A.7}$$

obtained setting $\eta_{i\rm B}$ =constant, where the ζ_i's are appropriate functions of the ξ's and ε. Equations (A.7) show that the functions $\widetilde{\eta}_i(\alpha,\xi,\varepsilon)$ are analytic in α at $\varepsilon\neq 0$. The α-structures of $\overline{\beta}_\alpha(\alpha,\eta_i)$ and $\overline{\beta}_i(\alpha,\eta_i)$ allow us to draw the same conclusion directly from (A.6).

Similarly, define

$$\widetilde{Z}_\alpha(\alpha,\xi,\varepsilon)\equiv\overline{Z}_\alpha(\alpha,\widetilde{\eta}_i(\alpha,\xi,\varepsilon),\varepsilon), \qquad \widetilde{\Delta}_i(\alpha,\xi,\varepsilon)\equiv\Delta_i(\alpha,\widetilde{\eta}_i(\alpha,\xi,\varepsilon),\varepsilon).$$

Written in this form, the renormalization constants satisfy ordinary first-order differential equations, obtained differentiating (A.5):

$$\frac{\mathrm{d}\ln\widetilde{Z}_\alpha}{\mathrm{d}\alpha}=-\frac{1}{\alpha}\frac{\overline{\beta}_\alpha(\alpha,\widetilde{\eta}_i)}{\overline{\beta}_\alpha(\alpha,\widetilde{\eta}_i)-\varepsilon\alpha}, \qquad \frac{\mathrm{d}\left(\alpha\widetilde{\Delta}_i\right)}{\mathrm{d}\alpha}=-\frac{\overline{\beta}_i(\alpha,\widetilde{\eta}_i)}{\overline{\beta}_\alpha(\alpha,\widetilde{\eta}_i)-\varepsilon\alpha}. \tag{A.8}$$

Integrate these equations, with the initial conditions

$$\widetilde{Z}_\alpha(0,\xi,\varepsilon)=1, \qquad \widetilde{\Delta}_i(0,\xi,\varepsilon)<\infty. \tag{A.9}$$

Such initial conditions are ensured by the α-structure proved above and the α-analyticity of $\widetilde{\eta}_i(\alpha,\xi,\varepsilon)$ at $\varepsilon\neq 0$. Observe that the solutions $\alpha\widetilde{\Delta}_i$ are immediate to find:

$$\alpha\widetilde{\Delta}_i(\alpha,\xi,\varepsilon)=\widetilde{\eta}_i(0,\xi,\varepsilon)-\widetilde{\eta}_i(\alpha,\xi,\varepsilon).$$

Inverting the functions $\widetilde{\eta}_i(\alpha,\xi,\varepsilon)$, write the ξ_i's as RG-invariant functions of the couplings and ε:

$$\xi_i=\xi_i(\alpha,\eta_j,\varepsilon). \tag{A.10}$$

Next, inserting (A.10) into $\widetilde{Z}_\alpha(\alpha,\xi,\varepsilon)$ and $\alpha\widetilde{\Delta}_i(\alpha,\xi,\varepsilon)$, $\overline{Z}_\alpha(\alpha,\eta_i,\varepsilon)$ and $\alpha\Delta_i(\alpha,\eta_i,\varepsilon)$ are obtained. Finally, using $\eta_i=\alpha^{-p_i}\lambda_i$ the renormalization constants

$$Z_\alpha(\alpha,\lambda_i,\varepsilon), \qquad Z_i(\alpha,\lambda_i,\varepsilon).$$

are successfully reconstructed.

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